Aharonov - Bohm media

A-B Phys. Rev. 115 (1959) 485

1. The problem

2. Complex coordinates

3. Single flux

4. 2 fluxes

5. flux lattice

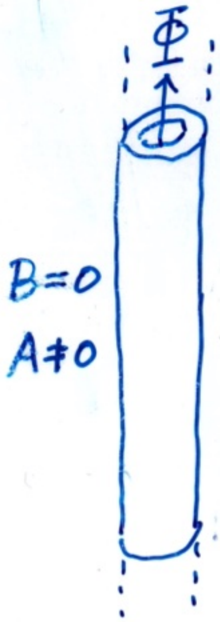
6. Presence of a magnetic field

$$m\dot{\vec{v}} = \frac{e}{c}(\vec{v}\times\vec{B})$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{B} = (0, 0, B_0)$$

$$\vec{A} = \frac{B_0}{2}(-y, x, 0)$$



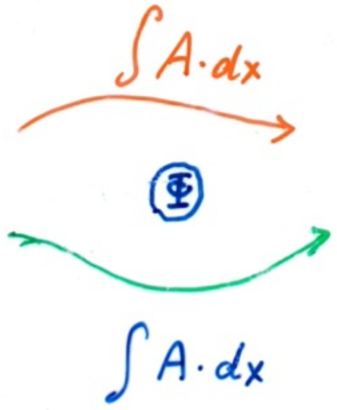
Flux $\Phi = B_0 \times \pi a^2$

$$\vec{A} = \frac{\Phi}{2\pi}\left(\frac{-y}{r^2}, \frac{x}{r^2}, 0\right)$$

$$(r \gg a)$$

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$$

$$E\psi = H\psi, \quad \vec{p} = -i\hbar\vec{\nabla}$$

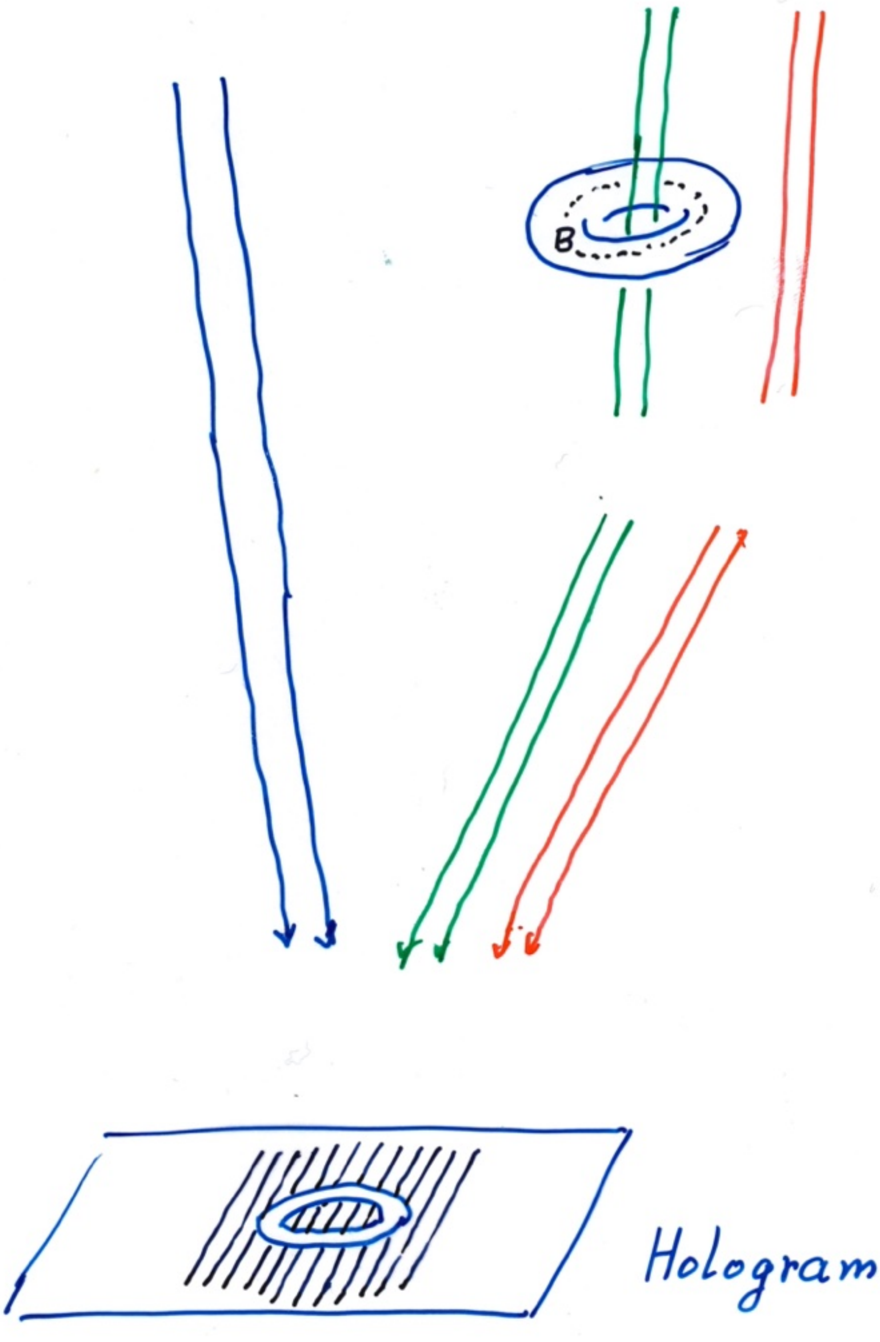


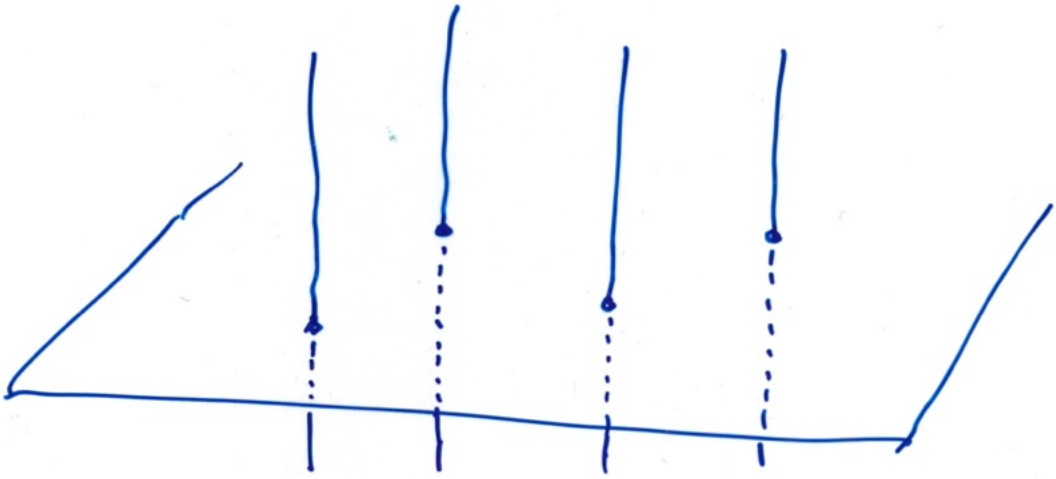
$$\psi = e^{\frac{i}{\hbar}S} \qquad p\psi = S'\psi$$

$$p = \frac{e}{c}A + \sqrt{2mE} = S'$$

$$S = \sqrt{2mE}\,x + \int \frac{e}{c}A\,dx$$

A. Tonomura et al. Phys. Rev. Lett. 56 (1986) 793





Flux quantization

$$H = \frac{1}{2m}(\vec{p} - e^*\vec{A})^2 = 0$$

$$\rightarrow \quad \vec{p} - e^*\vec{A} = 0$$

$$\rightarrow \; nh = \oint p \cdot dq = e^*\oint A \cdot dq = \Phi$$

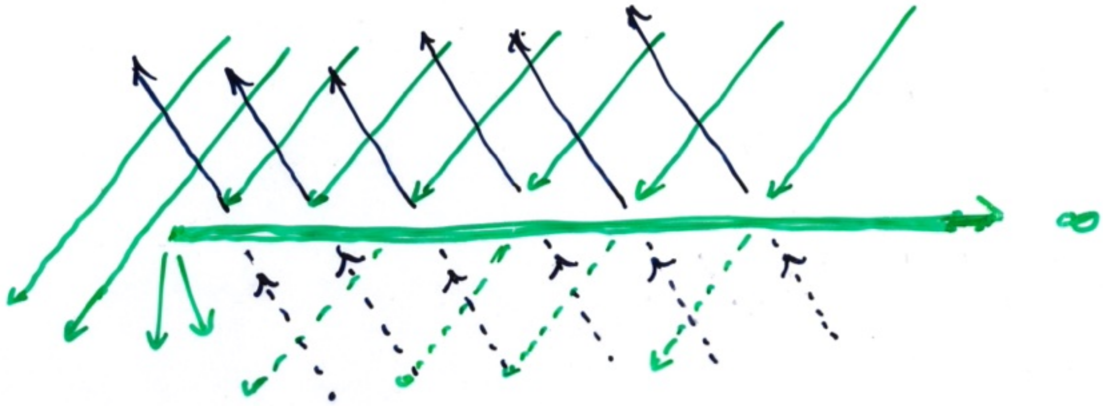
$$(e^*A_x, e^*A_y) = \frac{\alpha}{r^2}(-y, x) \quad , \; e^*\Phi = 2\pi\alpha$$

$$\alpha = 0, \pm 1, \pm 2, \cdots \qquad (\hbar = 1)$$

superconductors : $\quad e^* = 2e \qquad \Phi = \frac{2\pi n}{2e}$

Use of complex coordinates

A. Sommerfeld "Optics" (Thesis, 1896)



complex coordinates

$$x \pm iy = z, \bar{z}$$

$$A_x \mp iA_y = A_z, A_{\bar{z}}$$

$$D_z = \frac{\partial}{\partial z} - \frac{\alpha}{2z}, \quad D_{\bar{z}} = \frac{\partial}{\partial \bar{z}} + \frac{\alpha}{2\bar{z}}$$

$$H\Psi = \{D_z, D_{\bar{z}}\}_+ / m\Psi = E\Psi$$

$$\to \quad (D_z D_{\bar{z}} + E)\Psi = 0 \qquad (2mE \to E)$$

$$\Psi = G(z,\bar{z})\psi, \qquad G = (z/\bar{z})^{\alpha/2}$$

$$\left(\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + E\right)\psi = 0 \qquad (\bar{z} \neq z^*)$$

$$\left.\begin{array}{l} \psi(\theta = 2\pi) = e^{2\pi i\alpha}\psi(0) \\ \psi(r=0) = 0 \\ \psi(r=\infty) < \infty \end{array}\right\} \text{"on shell"} \quad \bar{z} = z^*$$

time reversal : $z \leftrightarrow \bar{z}, \quad \alpha \to -\alpha$

1. $E = 0$, $\partial_z \partial_{\bar{z}} \psi = 0$

$$\rightarrow \quad \psi = f(z) + g(\bar{z})$$

$$\rightarrow \quad \psi = \prod_i (z - b_i)^{-\alpha_i} P(z)$$

$$\text{or} \quad \prod_i (\bar{z} - \bar{b}_i)^{\alpha_i} P(\bar{z})$$

Non-integer phases $\{\alpha_i\}$ represented by z or $\bar{z}$ only. Mixed representation not possible.

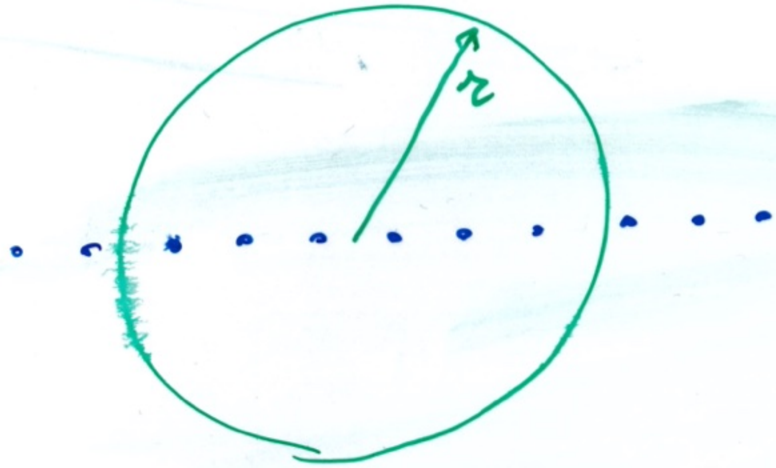
E.g. $\alpha_1 > 0$ $\quad -\alpha_2 < 0$

$$(z - b_1)^{\alpha_1} \qquad (z - b_2)^{-\alpha_2 + n} \qquad n - \alpha_2 > 0$$

$$\times \quad (\bar{z} - \bar{b}_2)^{\alpha_2}$$

The same situation persists for $E > 0$.

Linear chain



$$\psi \sim \exp(il\theta + ikz)$$

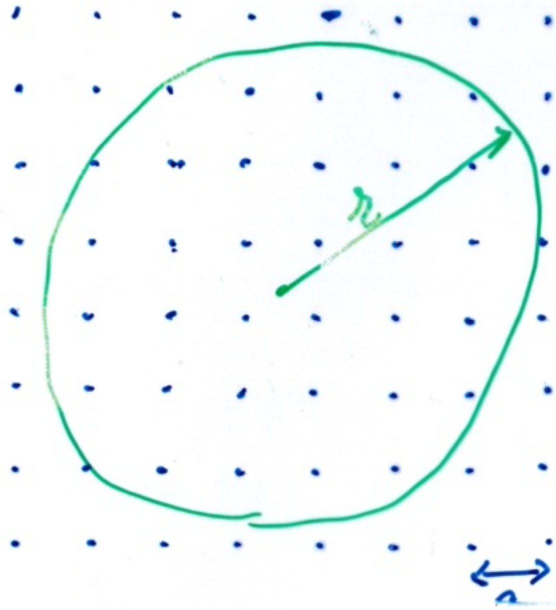
$$l \simeq \alpha(2r/a) \qquad \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \sim -4\alpha^2/a^2$$

$$k \sim \pm\sqrt{E - 4\alpha^2/a^2}$$

$\rightarrow \quad E < 4\alpha^2/a^2 \qquad$ damped

$\qquad > 4\alpha^2/a^2 \qquad$ transparent

2. Flux lattice (flux gas)



$$\Phi \approx \alpha \pi r^2 / a^2$$

$$\psi \sim e^{il\theta + iR(r)}, \qquad l \sim \alpha \pi r^2 / a^2$$

$$p^2 \sim -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \sim R'^2 + \frac{\alpha^2 \pi^2 r^2}{a^4} = E$$

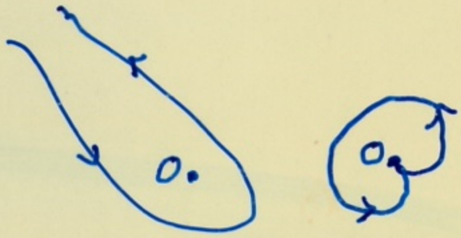
$$R' \sim \pm i \left(E - \frac{\alpha^2 \pi^2 r^2}{a^4}\right)^{1/2} \sim \pm \frac{\alpha \pi r}{a^2}$$

$$R \sim \pm \frac{1}{2} \alpha \pi r^2 / a^2$$

$$\psi \sim e^{i \alpha \pi r^2 / a^2 \theta \pm \frac{1}{2} \alpha \pi r^2 / a^2}$$

Single flux

$$\psi = \int_C e^{zt - \frac{\bar{z}E}{t}} f(t)\, dt \qquad C: \mathrm{Re}(zt) < 0$$



$$f(t) = t^{\alpha - n - 1} g(t)$$

$\psi(0) = 0 \rightarrow \quad n - \alpha > 0$

z-type solution $(1,0)$

$n - \alpha < 0 \rightarrow$ $\bar{z}$-type solution $(0,1)$

$E = 0$:

$$\psi \sim z^{-\alpha} g(z) \quad \text{or} \quad \bar{z}^{\alpha} g(\bar{z})$$

many fluxes

$$\psi \sim (z - b_1)^{-\alpha_1} (z - b_2)^{-\alpha_2} \cdots$$

$$\text{or} \quad (\bar{z} - \bar{b}_1)^{\alpha_1} (\bar{z} - \bar{b}_2)^{\alpha_2} \cdots$$

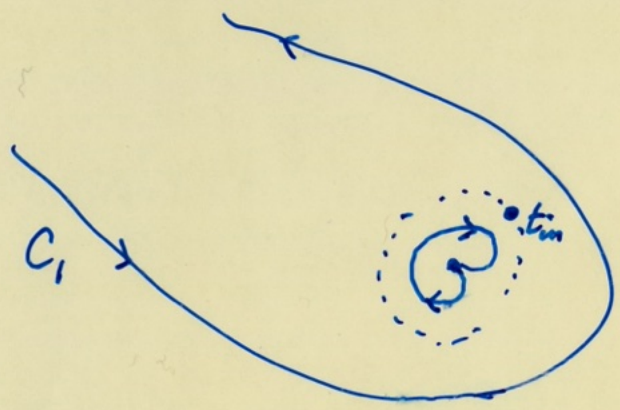
$E \neq 0$,

$$\Psi \sim (z/\bar{z})^{\alpha/2} J_{|n-\alpha|}\left(2\sqrt{E z \bar{z}}\right)$$

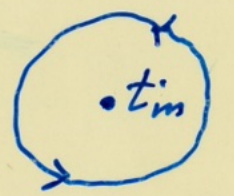
Scattering

$$\psi = \int_{C_1+C_2} e^{zt - \frac{\bar{z}E}{t}} \, t^{\alpha} \frac{1}{t-t_{in}} \, dt$$

$$|t_{in}| = \sqrt{E} \,, \qquad 0 > \alpha > -1$$



$$\alpha = 0 \;\rightarrow\; \psi = \psi_{in}$$



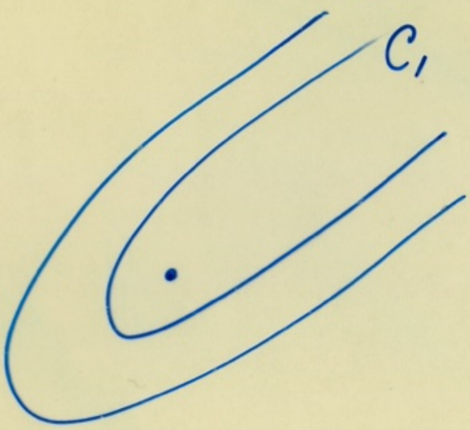
Asymptotics : integrand $\equiv e^{S(t)}$

$$\frac{\partial S}{\partial t} = 0 \;:\quad z + \frac{\bar{z}E}{t_0^2} + \ln f(t)' = 0$$

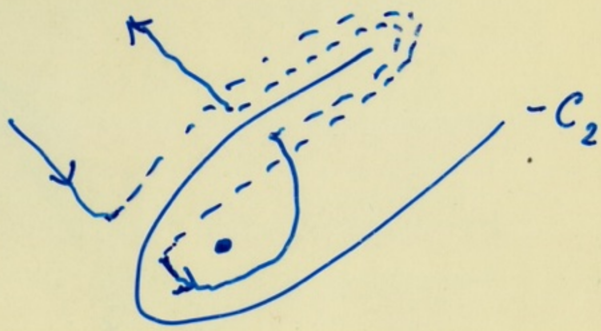
$$\frac{d\sigma}{d\Omega} \sim \frac{1}{\left(\sin\frac{\theta}{2}\right)^2}$$

2 fluxes

$$\psi = \iint_{C=(C_1,C_2)+(C_2,C_1)} e^{z_1 t_1 + z_2 t_2 - \frac{\bar{z}E}{t_1+t_2}}\, t_1^{\alpha_1 - 1}\, t_2^{\alpha_2 - 1}\, dt_1\, dt_2$$

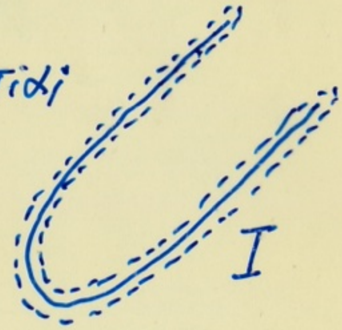


$t_1 = -t_2$

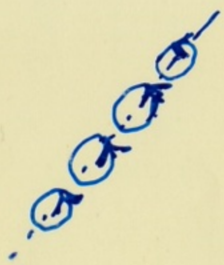


Monodromy relations

$$M_1 \psi = \eta_1 \psi + I_1, \quad \eta_i = e^{2\pi i \alpha_i}$$
$$M_2 \psi = \eta_2 \psi + I_2,$$



$$I_i = (1-\eta_i) I$$



Abelian monodromy

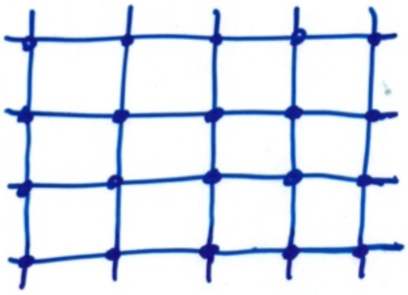
$$M_1 M_2 \psi = M_2 M_1 \psi \quad \text{if} \quad I_i = (1-\eta_i) I$$

→ monodromy projection

$$P = \frac{(1-M_1)}{1-\eta_1} = \left(\frac{1-M_2}{1-\eta_2}\right)$$

$$\psi_P = P\psi, \qquad M_i \psi_P = \eta_i \psi_P$$

Flux lattice



$$z_i = z - b_i$$

$$b_i = n + mi$$

$$A_z = \frac{\alpha}{2}\sum\left(\frac{1}{z-b_i} + \frac{1}{b_i} + \frac{z}{b_i^2}\right) = \frac{\alpha}{2}\zeta(z)$$

$$A_{\bar z} = -\frac{\alpha}{2}\sum\left(\frac{1}{\bar z-\bar b_i} + \frac{1}{\bar b_i} + \frac{\bar z}{\bar b_i^2}\right) = -\frac{\alpha}{2}\zeta(\bar z)$$

$$G(z,\bar z) = \exp\left(\frac{\alpha}{2}\int \zeta\, dz\right)\exp\left(-\frac{\alpha}{2}\int\zeta(\bar z)\, d\bar z\right)$$

$$= \left(\frac{\sigma(z)}{\sigma(\bar z)}\right)^{\alpha/2}$$

$$\sigma(z) = \prod_i\left[(z-b_i)\, e^{\frac{z}{b_i}+\frac{z^2}{2b_i^2}}\right]$$

$$S'_{t_i} = 0 : \quad z_i + \frac{\bar{z}E}{T^2} + \frac{\alpha}{t_i} - \frac{N}{T}$$

$$t_i = -\alpha \Big/ \left(z_i + \frac{\bar{z}E}{T^2} - \frac{N}{T}\right)$$

$$= -\alpha / (w - b_i)$$

$$= -\alpha / w_i$$

$$w = z + \frac{\bar{z}E}{T^2}$$

$$S'_{g_i} = 0 : \quad g_i = -\alpha\,(w + b_i)/b_i^2$$

$$\rightarrow T = \sum (t_i + g_i)$$

$$= -\alpha \sum \left(\frac{1}{w_i} + \frac{w + b_i}{b_i^2} \right) = -\alpha\, \zeta(w)$$

$$\det\left(S_{t_i t_k}\right) \det\left(S_{g_i g_k}\right) = dz/dw$$

$$\psi \sim \exp\left[2E\bar{z}/\alpha\zeta(w) \right] \sigma(w)^{-\alpha}\, \zeta(w)^{-N}$$

$$\times \left(\frac{dw}{dz}\right)^{-1/2}$$

鞍点法で必要な引算項を出す方法

$$\psi = \int \cdots \int \prod \frac{dt_i}{t_i} \prod dq_i \; e^{S}$$

$$\begin{aligned} e^{S} = \exp\Big[& \sum q_i^2 b_i^2 / 2\alpha + \sum (z - b_i) t_i \\ & + \sum (z + b_i) q_i \\ & - zE \Big/ \sum (t_i + q_i) \Big] \\ & \times \prod t_i^{\alpha} \left[\sum (t_i + q_i) \right]^{-N} \end{aligned}$$

Saddle point approximation

Lattice QFT analogy :

$$\int \prod d\phi_i \; e^{S(\phi_i)} \qquad \phi_i = t_i, q_i$$

Weierstrass elliptic functions

$$\wp(z) = \sum{}' \left(\frac{1}{z_i^2} - \frac{1}{b_i^2} \right)$$

$$\zeta(z) = -\int \wp(z)\,dz = \sum{}' \left(\frac{1}{z_i} + \frac{z+b_i}{b_i^2} \right)$$

$$\sigma(z) = \exp\left(\int \zeta(z)\,dz \right) = \prod \left(z_i \cdot e^{z^2/2b_i^2 + z/b_i} \right)$$

$$\wp(z+\Omega_{mn}) = \wp(z), \qquad \Omega_{mn} = m\Omega + n\Omega'$$

$$\zeta(z+\Omega_{mn}) = \zeta(z) + m\eta + n\eta'$$

$$\eta\Omega' - \eta'\Omega = 2\pi i$$

$$\Omega = 1, \quad \Omega' = i \;\rightarrow\; \eta = \pi, \quad \eta' = -\pi i$$

$$\sigma(z+\Omega_{mn}) \sim \sigma(w)\, e^{\pi|z|^2}$$

Presense of a magnetic field

$$A_z = \gamma\bar{z}/2 \,, \qquad A_{\bar{z}} = -\gamma z/2$$

$$\left(\frac{\alpha}{z} \to \frac{\alpha\bar{z}}{z\bar{z}} \to \frac{\alpha\bar{z}}{\rho^2} \,, \quad \frac{-\alpha}{\bar{z}} \to \frac{-\alpha z}{z\bar{z}} \to \frac{-\alpha z}{\rho^2} \right)$$

$$\left(\left\{ (\partial_z - \gamma\tfrac{\bar{z}}{2}), (\partial_{\bar{z}} + \gamma z/2) \right\}_+ /2 + E \right) \Psi =$$

$$= \left[(\partial_z - \gamma\bar{z}/2)(\partial_{\bar{z}} + \gamma z/2) + (E - \gamma/2) \right] \Psi = 0$$

$$\Psi = \exp(-\gamma z\bar{z}/2)\,\psi \qquad (\gamma > 0)$$

$$\left[(\partial_z - \gamma\bar{z})\,\partial_{\bar{z}} + E - \gamma/2 \right] \psi = 0$$

$$\psi_t = e^{zt}\,(t - \gamma\bar{z})^{E/\gamma - \frac{1}{2}}$$

t: center of Larmor orbit

$$\to E/\gamma - \tfrac{1}{2} = 0, 1, 2, \cdots$$

1. $\psi_n = e^{zt}(t - \bar{z}\gamma)^n$

2. $\psi_{n,l} = \int \psi_n\, t^{-l-1}\, dt \qquad l \geq 0$

3. $\psi = e^{zt}(t - \bar{z}\gamma)^n / t^n = e^{zt}\left(1 - \frac{\bar{z}\gamma}{t}\right)^{\frac{E}{\gamma} - \frac{1}{2}}$

$$\lim_{\gamma \to \infty} = e^{zt - \bar{z}E/t}$$

A flux + magneti field

1. $\psi_{n+\alpha,t} = e^{zt}(t-\bar{z}\gamma)^{n+\alpha}$ $\quad n+\alpha \geq 0$

$$\frac{E}{\gamma} - \frac{1}{2} = n+\alpha \geqslant 0$$

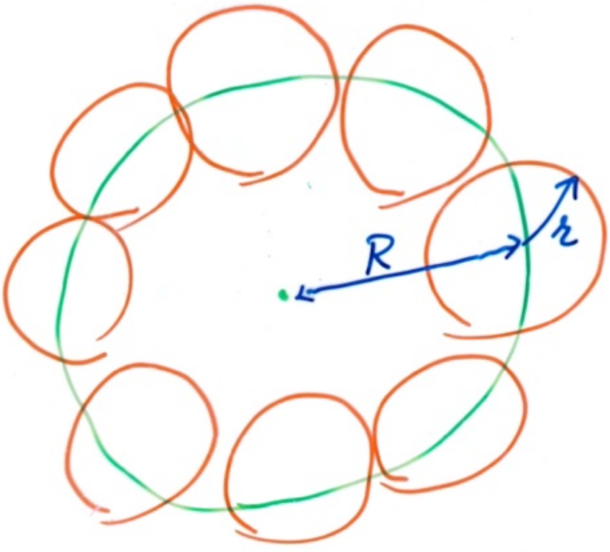
2. $\psi_{n+\alpha,\ell} = \oint \psi_{n+\alpha,t}\, t^{-\ell-1} dt$

$0 < \ell \leq \ell_0 = [n+\alpha]$ $\qquad \frac{E}{\gamma} - \frac{1}{2} = n+\alpha$

3. $\psi_{n,\ell} = \int_c \psi_{n,t}\, t^{-\ell+\alpha-1} dt$

$\ell > \ell_0$ $\qquad \frac{E}{\gamma} - \frac{1}{2} = n$

(R. R. Lewis, Phys. Rev. A28 (83) 1228)

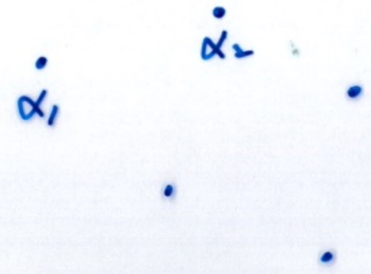


$R \sim (n/\gamma)^{1/2}$

$r \sim \ell/(\gamma n)^{1/2}$

$$\frac{R}{r} = \frac{n}{\ell}$$

Many fluxes + magnetic field.



1. Unshifted levels $\qquad E/\gamma - \frac{1}{2} = n$
2. Shifted levels $\qquad E/\gamma - \frac{1}{2} = n + \alpha_i$

$$i = 1, \cdots$$

but not $\quad n + \Sigma \alpha_i$

Flux lattice

$$\psi \sim \exp[-(\pi + \gamma\alpha) z\bar{z}/2]$$

$$\psi = \int e^{\Sigma z_i t_i - \frac{\bar{z} E}{T}} \left(\Pi t_i^{\alpha - 1} \right) T^{-N} dt_i$$

$$T = \Sigma t_i , \qquad N = 1 , \quad 0 < \alpha < 1$$

$$\phi = e^{-\bar{z} E / T}$$

$$\left(\frac{\partial}{\partial t} - \nabla_d^2 \right) \chi = \delta , \qquad \chi = e^{-x^2/t} / t^{d/2}$$

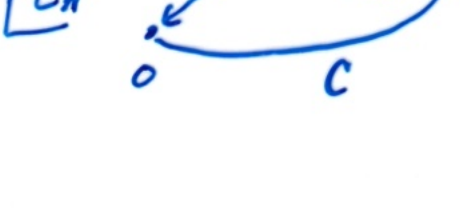
$$\chi = \int_0^\infty d\lambda \, e^{-\lambda t} J_{d/2-1} (2x\sqrt{\lambda}) (x\sqrt{\lambda})^{1-d/2}$$

$$\rightarrow \quad \phi = \int_0^\infty e^{-\lambda T} J_0 (2\sqrt{\lambda \bar{z} E}) \, d\lambda \quad \left(Re(\lambda T) > 0 \right)$$

$$\psi = \int_0^\infty \left[\int_{C_i} e^{\Sigma (z_i - \lambda) t_i} \left(\Pi t_i^{\alpha-1} dt_i \right) \right] J_0 (2\sqrt{\lambda \bar{z} E}) \, d\lambda$$

Let $z - b_n = z_n$ small

$$\int_{C_n} e^{(z_n - \lambda) t_n} t_n^{\alpha - 1} dt_n = (z_n - \lambda)^{-\alpha} (\alpha - 1)!$$



$$\int_0^\infty \rightarrow \int_C$$

$$J_0(2\sqrt{\lambda \bar{z} E}) = \sum_m \frac{(-\lambda \bar{z} E)^m}{m!^2}$$

$$\int_0^{z_n} (z_n - \lambda)^{-\alpha} \lambda^m \, d\lambda = z_n^{m-\alpha+1} \int_0^1 (1-s)^{-\alpha} s^m \, ds$$

$$= z_n^{m-\alpha+1} B(-\alpha+1, m+1) = z_n^{m-\alpha+1} \frac{(-\alpha)! \, m!}{(m+1-\alpha)!}$$

$$\int_0^{z_n} (z_n - \lambda)^{-\alpha} J_0(2\sqrt{\lambda \bar{z} E}) \, d\lambda = (-\alpha)! \, z_n^{1-\alpha} \sum \frac{(-\cancel{\lambda} z_n \bar{z} E)^m}{m! \, (m+1-\alpha)!}$$

$$= \alpha! \, J_{1-\alpha}(2\sqrt{z_n \bar{z} E}) \left(\frac{z_n}{\bar{z} E}\right)^{\frac{1-\alpha}{2}}$$

$$\int e^{(z_i - \lambda) t_i} \prod t_i^{\alpha-1} \, dt_i \Rightarrow \prod (z_i - \lambda)^{-\alpha} = \sigma(z-\lambda)^{-\alpha}$$

$$\lambda : 0 \to z_n \qquad Re\,(z_i - \lambda) t_i \sim Re\,(z_i - z_n) t_i \sim Re\, z_i t_i$$

$$(i \neq n)$$

$$\boxed{\psi \cong \int_{0 \cdots (z_n)_+} \sigma(z-\lambda)^{-\alpha} J_0(2\sqrt{\lambda \bar{z} E}) \, d\lambda}$$

Saddle point approximation

$$J_0(\sqrt{\lambda \bar{z} E}) \sim \frac{1}{\sqrt{\pi \lambda \bar{z} E}} \sin\left(2\sqrt{\lambda \bar{z} E} - \frac{\pi}{4}\right)$$

$$= -\frac{1}{2\sqrt{\pi \lambda \bar{z} E}} \left(\sqrt{i}\, e^{2i\sqrt{\lambda \bar{z} E}} + h.c.\right)$$

$$\psi \to \int \sigma(z-\lambda)^{-\alpha} e^{\pm 2i\sqrt{\lambda \bar{z} E}} \frac{d\lambda}{\sqrt{\lambda \bar{z} E}}$$

$$\frac{\partial}{\partial \lambda} \ln (z-\lambda)^{-\alpha} \pm i\sqrt{\frac{\bar{z}E}{\lambda}} \sim 0$$

$$\alpha \zeta(z-\lambda) \pm \quad " \quad \sim 0$$

$$\lambda = \frac{-\bar{z}E}{(\alpha \zeta(z-\lambda))^2}$$

$$\to \quad \lambda = z - w, \qquad w = z + \frac{\bar{z}E}{(\alpha \zeta(w))^2}$$