Bohm-Aharonov effect.

$$A_\theta = \alpha \quad \rightarrow \quad H = \frac{1}{2m}\left(p_\rho^2 + \frac{(L+\alpha)^2}{\rho^2}\right) \qquad (2\text{-dim.})$$

$$\psi = \sum_n f_n(\rho)\, e^{in\theta} \quad \rightarrow \quad f_n = J_{n+\alpha}(k\rho)$$

$f(\rho)$ regular at $\rho = 0$ $\rightarrow$ $J_{|n+\alpha|}(k\rho) \sim (k\rho)^{|n+\alpha|}$

Remark: ψ is one-valued, but A_θ is not. Is this all right?

What is a "plane wave"?

Usual plane wave $e^{ik\rho\cos\theta}$

$$\int e^{ik\rho\cos\theta}\, e^{-in\theta}\, d\theta = J_n(k\rho)$$

$$\rightarrow \quad e^{ik\rho\cos\theta} = \sum_n J_n(k\rho)\, e^{in\theta}$$

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\, \cos\left(z - \frac{2\nu+1}{4}\pi\right)$$

ψ cannot approach a pure plane wave.

Change of zero's of $J_{n+\alpha}(\underset{\underset{x_i}{\wr}}{k_i R})$ as a fn of α.

$$dx\frac{\partial J_n(x)}{\partial x} + d\alpha\frac{\partial J_n(x)}{\partial \cancel{\alpha} n} = 0$$ but $\frac{\partial J_n}{\partial n}$ not easily expressible

Asymptotically, $J_\nu \sim \frac{\sqrt{2}}{\sqrt{\pi x}} \cos\left(\overset{x}{\cancel{2}} - \frac{2\nu+1}{4}\pi\right)$

$$x_{\underset{m}{\cancel{\nu}}} = m\pi + \frac{1}{2} + \frac{2\nu+1}{4}\pi \qquad \nu = n \pm \alpha$$

The ± changes cancel

What does this have to do with ~~ca~~ our confinement problem?

Q. My most naive question: In 2-dim, the cross section formula would be

$$d\sigma = \frac{2\pi}{v}|V|^2 \frac{d^2p}{(2\pi)^2}\delta(E-E_0) \rightarrow \frac{1}{2\pi v}|V|^2 \frac{p\,d\theta}{\frac{dE}{dp}} = \frac{1}{2\pi}|V|^2\frac{p}{v^2}\,d\theta$$

$\rightarrow \infty$ for $v \rightarrow 0$ if $|V|^2 \rightarrow$ finite?

Probably this is true: A $\overset{\text{hard core}}{\wedge}$ potential, radius a:

S-wave sol. $J_0 \sim 1$, $N_0 \sim \ln k\rho$ $\cancel{k\rho}$

$$\psi = \cancel{a}J_0 + \overset{\varepsilon}{\cancel{b}}N_0 \quad \text{such that} \quad \psi(a) = 0$$

$$\text{or} \quad 1 + \varepsilon \ln ka = 0 \rightarrow \varepsilon = -1/\ln ka$$

$$\sigma \sim \frac{1}{k}\varepsilon^2 = -\frac{1}{k}\frac{1}{(\ln ka)^2} \rightarrow \infty \text{ as } k \rightarrow 0$$

This happens only for $m = 0$ state.

On the other hand, for 3-dim.,

$$\psi \sim 1 + \varepsilon/k\rho \quad \rightarrow \quad 1 + \varepsilon/ka = 0$$

$$\varepsilon = ka, \qquad \sigma \approx \varepsilon^2/k^2 = a^2 \text{ finite.}$$

Back to a Bohm-Aharonov gas.

An array of eddy centers with various strengths.

$$(A_x, A_y) = \vec{A} \qquad A_i = \varepsilon_{ij}\,\partial_j \chi$$

or else $\partial_i \phi$

$$\chi = \sum_n \ln(\vec{r}-\vec{r}_n)^2 \cdot \alpha_n/2 \qquad \rightarrow \text{needs a scale parameter}$$

$$\phi = \sum_n \tan^{-1}(x-x_n)/(y-y_n) \cdot \alpha_n \qquad \text{up to an additive const.}$$

These are divergent.

I think the best way to handle it is to use functional integration.

In 2-dim., proper time form.

$$L = \int \left(\frac{dr}{dt}\right)^2 dt + i\int A \cdot \dot{r}\, dt$$

Fourier decomp.

$$\vec{r} = \vec{v}t + \sum \vec{r}_n \sin \frac{n\pi}{T} t \; \cancel{\sin m\pi y} \quad , \quad \vec{v} = \vec{R}/T$$

$$\dot{\vec{r}} = \vec{v} + \sum \frac{n\pi}{T} \vec{r}_n \cos \frac{n\pi}{T} t$$

$$\int \dot{r}^2 = \vec{v}^2 T + \sum \left(\frac{n\pi}{T}\right)^2 \vec{r}_n^{\,2} \cdot \frac{T}{2}$$

Weights : $\Delta \times \left(x(t_1)^2 + x(t_2)^2 + \cdots \quad x(t_N)^2\right) \qquad \Delta = T/N$

$$\rightarrow \int x^2\, dt = \sum \cancel{\left(\frac{n\pi}{T}\right)}^2 \cdot \frac{T}{2}\, r_n^2 + \int (\vec{v}t)^2\, dt \qquad ?$$

We should take $\prod_n \frac{dx(t_n)}{\sqrt{\Delta}}$ but why?

$$\int \exp\left[-\frac{(n\pi)^2}{2T} r_n^2\right] dr_n = \sqrt{\pi}\cdot\frac{\sqrt{2T}}{n\pi}$$

$$\times \frac{1}{\sqrt{\Delta}} \rightarrow \frac{\sqrt{N}}{n}$$

Further Jacobian from $dx(t_i) \rightarrow dr_n$

$$dx(t_i) = \sqrt{\frac{T}{\Delta}}\, dr_n = \sqrt{N}\, dr_n$$

So $\int \exp[\quad] \, dr_n \sqrt{N} \rightarrow \frac{N}{n}$ This is O.K.

So the proper weight for dr_n is $\sqrt{\frac{N}{\Delta}}\, dr_n = N\frac{dr_n}{\sqrt{T}}$

Use of complex variables. $x + iy = z$. $\nabla^2 = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$

The potential can be eliminated by gauge transf. ϕ:

$$\psi = \prod_{0}\left(\frac{z - z_0}{\bar{z} - \bar{z}_0}\right)^{\alpha_n/2} \times f(z, \bar{z})$$

$$\left(\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + m^2\right) f = 0$$

and ψ must be one-valued. So near $z = z_n$,

$$f \sim (\bar{z} - \bar{z}_n)^{\alpha_n + K_n} \quad \text{if} \quad \alpha_n > 0. \qquad K_n \geqslant 0$$

$$\sim (z - z_n)^{-\alpha_n + K_n} \quad \text{if} \quad \alpha_n < 0.$$

$$\text{so} \quad \left(\frac{\alpha_n + K_n}{\bar{z} - \bar{z}_n}\frac{\partial}{\partial z} + m^2\right) f \sim 0 \qquad \text{if } \alpha_n > 0$$

$$\rightarrow f \sim (\bar{z} - \bar{z}_n)^{\alpha_n + K_n} \exp\left[-\frac{m^2}{\alpha_n + K_n}(z - \bar{z}_n)(z - z_n)\right] g \quad \text{near } z = z_n \quad ?$$

$(z - z_0)$

Since I cannot make sense out of B-A, I go back to the original problem.

$I_{nt} = \int \mathcal{F}_{\mu\nu} \, d\sigma^{\mu\nu}$

Q. Entropy of monopoles. $\mathcal{F}_{\mu\nu} = \underline{\partial_\mu(\phi A_\nu)} - \underset{\underset{G_{\mu\nu}}{\parallel}}{\partial_\mu \phi \times \partial_\nu \phi \cdot \phi}$

$G_{\mu\nu}$ is the "potential" for monopole current:

$\partial_{(\mu} G_{\nu \underset{\lambda}{\rho})} = k_\rho$

Isn't it possible that $S \propto k_\rho k^\rho$? Which sign?

Call $k_\rho \to V_\rho = \partial_\lambda \tilde{G}_{\lambda\rho}$ Time derivative in space compts of V.

So $\mathcal{L} = -V_\rho V^\rho \to V_\rho V_\rho$ Euclidean?

No: $V_{oi} = \partial_{(k} G_{jo)} \to V_i^2$ changes sign.

Thus $S' \propto - V_\rho V_\rho$

We have to add a mass term $\sim G_{\mu\nu} G^{\mu\nu}$?

This is not gauge inv. so make it $\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}$?

Or use $G_{\mu\nu} \equiv D_\mu \phi \times D_\nu \phi \cdot \phi$, $V_\rho = D_\lambda \tilde{G}_{\cancel{j\alpha}\lambda\rho}$

First order form probably necessary.

$V_\rho = D_\lambda \tilde{W}_{\lambda\rho}$, $W_{\mu\nu} = D_\mu \phi \times D_\nu \phi \cdot \phi$

Duality of I' & I'' ?

I'': Monopoles are real, but not strings.

I': The strings $\sim \partial_\mu \partial_\nu u^+$ [illegible] u ~~con~~ $d\sigma^{\mu\nu}$. are felt by the sheet

$\sigma \rightarrow$ local form $u^+ D_\mu u j^\mu$. for charge.

Let the charge be represented by

$$j^\mu = \varepsilon^{\mu\nu\lambda\rho}\, \partial_\nu \vec{X} \times \partial_\lambda \vec{X} \cdot \partial_\rho \vec{X} = \varepsilon^{\mu\nu\lambda\rho}\, \partial_\nu [X \cdot \partial_\lambda X \times \partial_\rho X]$$

$$\text{or} \quad \varepsilon^{\mu\nu\lambda\rho}\, \partial_\nu (\partial_\lambda v^+ \partial_\rho v)$$

Contining on the Aharonov-Bohm problem.

$$A_x = \overset{2\pi\alpha}{\cancel{\alpha}}\, \theta(x)\, \delta(y) \quad \text{is O.K}$$

$$J_\mu = -\frac{i}{2} \psi \overleftrightarrow{D}_\mu \psi = -\frac{i}{2} \psi^\dagger \partial_\mu \psi - \psi^\dagger \psi A_\mu$$

This means that there is a cut along the $+x$ axis, and the phase will jump by $\overset{2\pi\alpha}{\cancel{\alpha}}$.

Sol. to $\left(\frac{\partial}{\partial z}\frac{\partial}{\partial z} + m^2\right) \psi = 0$:

$$\psi = \exp[ikz + ik'\bar{z}] \qquad kk' = m^2 \rightarrow k' = \frac{m^2}{k}$$

$$= \exp[ikz + i\frac{m^2}{k}\bar{z}]$$

Isotropic sol: $k = \frac{m^2}{k}$

We can superimpose different $k = |k| e^{i\varphi}$

$$z = \rho e^{i\theta} \qquad kz + k\bar{z} = \rho |k| \left(e^{i(\theta+\varphi)} + e^{-i(\theta+\varphi)}\right)$$

Let $$\psi = \frac{1}{2\pi}\int^{2\pi} \exp[2i\rho|k| \cos(\theta+\varphi)] \cdot e^{in\varphi}\, d\varphi$$

$$= \frac{1}{2\pi}\int^{0} \exp[2i\rho|k| \cos x] \exp[inx] \exp[-in\theta]\, dx$$

$$= \exp[i\frac{\pi}{2}n] J_n(2\rho|k|)\, e^{-in\theta}$$

To make a gap, choose $n \rightarrow n + \alpha$.

This fn is <u>not</u> $J_{n+\alpha}$.

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-z\sin\theta}\, e^{in\theta}\, d\theta$$

$$J_\nu(z) = -\frac{1}{2\pi i}\int_C \exp\left[\frac{z}{2}\left(\frac{1}{t}-t\right)\right](-t)^{-\nu-1}\, dt$$

$R z > 0$

$$= -\frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \int_C \exp\left[\frac{z^2}{4t}-t\right](-t)^{-\nu-1}\, dt$$

~~$\Re \nu > 1$~~

$$= \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \int_{c-\infty i}^{c+\infty i} \exp\left[t-\frac{z^2}{4t}\right] t^{-\nu-1}\, dt$$

$c > 0$

$\Re \nu > 1$.

April 19. Back to work!

Bohm - Aharonov continued.

Take the gauge $A_x = 0$

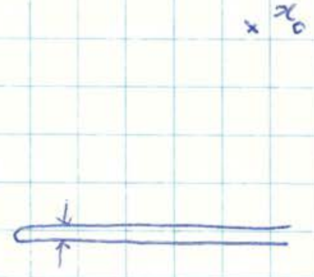
$$A_y = -\alpha\,\delta(y)\,\theta(x)$$

so the wave function jumps by a phase factor α along ~~across~~ the positive x axis.

$$(\nabla^2 + k^2)\psi = 0$$

$$(\nabla^2 + k^2) G = -\delta^2(\underline{x} - x_0) \quad G : \text{ Green's function}$$

$$\psi(x) = \int_D (G\nabla^2\psi - \nabla^2 G \cdot \psi)\, dv = \int \vec{\nabla}(G\vec{\nabla}\psi - \vec{\nabla}G\,\psi)\, dv$$



$$= \int_\Gamma (G\vec{\nabla}\psi - \vec{\nabla}G\,\psi)\, d\vec{S}$$

$$= -\int [G\,\partial_y\psi - \partial_y G\,\psi]_-^+\, dx$$

$$= \int [\partial_y G(\Delta\psi) - G(\Delta\partial_y\psi)]_-^+\, dx$$

$$\Delta\psi = \psi(y+0) - \psi(y-0)\Big|_{y=0} \Rightarrow (e^{i\alpha} - 1)\,\psi(y-0)\Big|_{y=0}$$

$$\Delta\partial_y\psi = \; ? \qquad \text{Maybe } = (e^{i\alpha} - 1)\,\partial_y\psi(y-0)_{y=0}$$

$$G(x) \propto H_0(k\rho) \sim \frac{1}{\sqrt{k\rho}}\, e^{ik\rho}$$

Perturbation theory: Let $\psi = \psi_0 + \varphi$

ψ_0 : no discontinuity. ~~$\Rightarrow \varphi = \psi_0 +$~~

$$\Rightarrow \psi_0 + \varphi = \cancel{\psi_0 +} \int_\Gamma (\partial_y G \, \Delta\varphi - G \, \Delta\partial_y \varphi) \, dx$$

Can one set $\Delta\varphi = (e^{i\alpha} - 1)\psi_0$?

$$\Delta\psi = (e^{i\alpha} - 1)\psi \qquad \psi_+ = \psi_0 + \varphi_+ \quad \psi_- = \psi_0 + \varphi_-$$

$$= \cancel{\psi_0 +} \Delta\varphi \qquad = e^{i\alpha}(\psi_0 + \varphi_-)$$

Compare this with the ordinary scat. problem:

$$\psi = \psi_0 + KV\psi$$

In this case $KV\psi$ is localized, so $\psi \sim \psi_0$

$$\Delta\varphi = (e^{i\alpha} - 1)(\psi_0 + \underline{\varphi}) \; \cancel{\Rightarrow \psi_0} \qquad \varphi_+ - \varphi_- = (e^{i\alpha} - 1)\psi_0 + (e^{i\alpha} - 1)\varphi_-$$

$$\varphi_+ = (e^{i\alpha} - 1)\psi_0 + e^{i\alpha}\varphi_-$$

Actually we ~~can~~ should take into acct the boundary term at ∞. Since G is $\overset{?}{\text{short}}$ range, the contribution from $\int_\Gamma \to 0$. Then this boundary term should = ψ_∞ or ψ_0. Then

$$\psi = \psi_0 + \int_\Gamma$$

$$G \sim \frac{e^{ik\rho}}{\sqrt{\rho k}} = O\left(\frac{1}{\sqrt{k\rho}}\right) \quad \text{compared to } \psi_0 . \quad ?$$

But Γ extends to ∞.

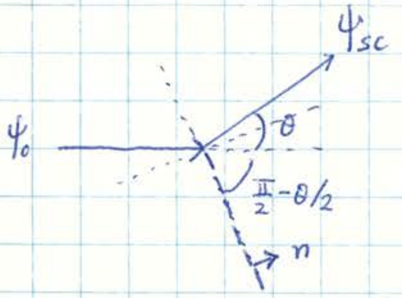
Including the contribution at ∞ :

$$\psi(x) = \int_{\Gamma} + \int_{\infty}$$

Q:1. $\int_{\infty} = \psi_0$?

Q.2. Can one deform Γ and make the best estimate (saddle pt)?

Actually we can make Γ simple & integrable.



Choose the cut to bisect the ψ_0 & ψ_{scat}.

$$\psi_0\,\partial_n G - G\,\partial_n\psi_0 = -2\cos\frac{\theta}{2}\,\psi_0 G \propto -\overset{k}{\underset{\wedge}{2}}\cos\frac{\theta}{2}\,e^{ikz}\,e^{ik\rho}$$

ds: along the cut $\qquad dz = \sin\frac{\theta}{2}\,ds = d\rho$

$$\Rightarrow \int ds\,(\psi_0\,\partial_n G - G\,\partial_n\psi_0) = -2\cos\frac{\theta}{2}\int_0^{\infty} e^{2iks\sin\frac{\theta}{2}}\,ds$$

$$= -i\cos\theta/_2 \,/\, \cancel{k}\sin\theta/_2$$

$$\psi_{scat} \sim -i\,\frac{1}{2\pi}\,\frac{1}{\sqrt{\rho}}\,\frac{\cos\theta/2}{\sin\theta/2}\,\psi_0\,(e^{i\alpha}-1) \quad ?$$

Green's fn $G(\rho)$ must be $\sim -\frac{1}{2\pi} \ln \rho$ for small ρ

$$= c\, H_0^{(1)}(k\rho)$$

$$H_0^{(1)}(z) = J_0 + i N_0 \sim i N_0 \sim \frac{2}{\pi}\left(\ln \frac{z}{2} + \gamma\right)$$

so $\quad \frac{2}{\pi} c\, i = -\frac{1}{2\pi} \qquad c = \frac{i}{4}$

Then $\quad H_0(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i(z - \pi/4)}$

$$c H_0(k\rho) \sim \frac{i}{4}\sqrt{\frac{2}{\pi k\rho}}\, e^{-i\pi/4}\, e^{ik\rho}$$

$$\psi_{sc} \sim (e^{i\alpha} - 1)(-i)\frac{\cos\theta/2}{\sin\theta/2} \cdot \frac{i}{4}\sqrt{\frac{2}{\pi k\rho}}\, e^{-i\pi/4}\, e^{ik\rho}\, \psi_0(0)$$

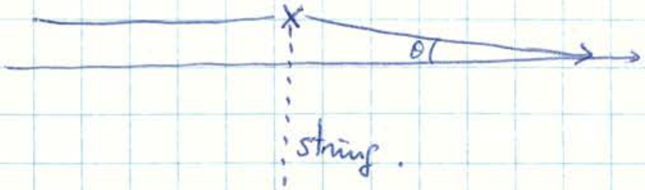
$$= i \sin\frac{\alpha}{2}\, \frac{\cos\theta/2}{\sin\theta/2}\, \frac{1}{\sqrt{2\pi k\rho}}\, e^{ik\rho}\, e^{-i\pi/4}\, \psi_0(0)$$

$\underset{\text{"Born"}}{d\sigma}$ = ~~$|\psi_{sc}|^2$~~ $\sin^2\frac{\alpha}{2}\, \cot^2\theta/2\, \frac{1}{2\pi k}\, d\theta$

According to A-B,

$$d\sigma = \sin^2\frac{\alpha}{2}\, \frac{1}{\sin^2\theta/2}\, \frac{1}{2\pi k}\, d\theta$$

Simple estimates :



The two paths differ by : $\Delta = k\rho\,(1-\cos\theta) - \alpha$ in phase.

→ $\Delta = \frac{1}{2}\,k\rho\,\cancel{\sin}\,\theta^2 - \alpha$

Interference max. at $\Delta = 0$, $k\rho\,\theta^2 = 2\alpha$,

This relation is strange. Utterly different from previous results ?

Well, we have $\psi_{sc} \sim \frac{\alpha}{\theta\sqrt{k\rho}}$

This is ~ 1 when $\sqrt{k\rho}\;\theta = \alpha$

The problem is similar to a semi-infinite screen. But what determines the diffraction width ? There is only one length scale : $\bar{\lambda}$.

Answer. It depends on ρ as well. Fresnel diffraction.

The parameter is precisely $k\rho\,\theta^2 = k\,x^2/\rho \sim 2\pi$

At $\rho \to \infty$, the scale is lost, and the angular dependence becomes "natural" : ~~no~~ indep of k.

Sommerfeld, Optics
p. 249 ff.

Formula for a staight edge.

$$U \cancel{\psi_{sc}} = -\frac{1+i}{4}\frac{1}{\sqrt{\pi k r}}\frac{1}{\cos\frac{\psi}{2}}e^{ikr}$$

$$|\overset{U}{\cancel{\psi_{sc}}}|^2 = \frac{1}{8}\frac{1}{\pi k r}\frac{1}{\cos^2\frac{\psi}{2}}$$

In fact, the ~~old~~ previous Born approx gave

$$\psi_{sc} = i\sin\frac{\alpha}{2}\frac{\cos\theta/2}{\sin\theta/2}e^{-i\pi/4}\frac{1}{\sqrt{2\pi k\rho}}e^{ik\rho}$$

Here $\sin\theta/2 \longleftrightarrow \cos\psi/2$, $e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$, $\cancel{k}\,\rho \longleftrightarrow r$

$$i\frac{(1-i)}{\sqrt{2}}\frac{1}{\sqrt{2\pi k\rho}} = \frac{1+i}{2\sqrt{\pi k\rho}}$$

So the correspondence is

$$-\frac{1}{2} \longleftrightarrow \sin\frac{\alpha}{2}$$

if $\cos\theta/2$ is ignored.

A-B effect can be larger than a semi-infinite wall!

No: The true formula was

$$u = U(\varphi-\alpha) + U(\varphi+\alpha)$$

$$\rightarrow \frac{1+i}{4\sqrt{\pi k r}}e^{ikr}\left(\frac{1}{\cos\frac{\varphi-\alpha}{2}} \cancel{\pm} \mp \frac{1}{e\cos\frac{\varphi+\alpha}{2}}\right)$$

$$\varphi-\alpha = \psi$$

Trial formula à la Sommerfeld.

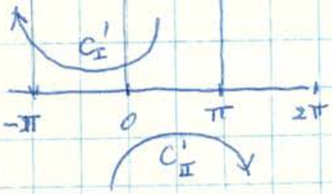
Consider $$\int_C e^{-ikr\cos(\varphi-\beta)} \frac{e^{ik\beta}}{e^{i\beta}-e^{i\alpha}}\, d\beta$$

$$= \int_{C'} e^{-ikr\cos\gamma} \frac{e^{ik(\gamma+\varphi)}}{e^{i(\gamma+\varphi)}-e^{i\alpha}}\, d\beta \qquad \gamma = \beta-\varphi$$

$$= \int_{C'} e^{-ikr\cos\gamma} \frac{e^{i\gamma k} e^{i(k-1)\varphi}}{e^{i\gamma}-e^{i(\alpha-\varphi)}}\, d\gamma \qquad k-1 : \text{AB flux.}$$

Contour C' : $\gamma \to \gamma + i\varepsilon$ $\quad \cos(\gamma+i\varepsilon) = \cos\gamma \,\mathrm{ch}\,\varepsilon - i\sin\gamma\,\mathrm{sh}\,\varepsilon$

$$C' = C'_I + C'_{II}$$



This fn is multivalued in φ

I.e. we have extended the space to a Riemann sheet.

$$\psi(\varphi+2\pi) = e^{2\pi i(k-1)}\psi(\varphi)$$

$e^{i\gamma}$ ~~$re^{i\beta} = z$~~ $\quad ke^{i\gamma} = w$

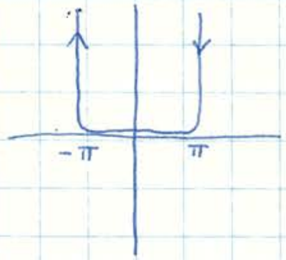
$\mathrm{Im}\,\gamma = \varepsilon > 0$ corresponds to $|w| > k$

$\phantom{\mathrm{Im}\,\gamma = \varepsilon} < 0$ $\qquad\qquad\qquad\qquad\quad < k$

Representations of Bessel fns.

$$J_\nu(z) = \frac{1}{2\pi i}\int_C e^{z\sinh t-\nu t}\,dt$$

$t \to -it$ $$-i\int_C e^{-iz\sin t+i\nu t}\,dt$$

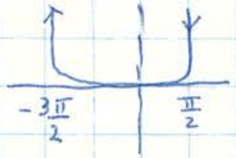


$\mathrm{Re}\, z > 0$

(Schläfli)

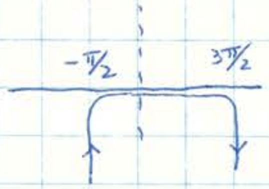
$t \to \frac{\pi}{2}+t$ $\quad \sin t \to +\cos t$

$$i e^{+\frac{\pi}{2}i\nu}\int e^{\mp iz\cos t+i\nu t}\,dt$$



$t \to \frac{\pi}{2}-t$ $\quad \sin t \to \cos t$

$$-i e^{\frac{\pi}{2}i\nu}\int e^{-iz\cos t-i\nu t}\,dt$$



Now $e^{ik(\gamma+\varphi)}\Big/\left(e^{i(\gamma+\varphi)}-e^{i\alpha}\right)$

$$= -e^{ik(\gamma+\varphi)-i\alpha}\Big/\left(1-e^{i(\gamma+\varphi-\alpha)}\right)$$

$$= -e^{ik(\gamma+\varphi)-i\alpha}\sum_{n=0}^{\infty} e^{in(\gamma+\varphi-\alpha)}$$ as a formal expansion

$$= -e^{-i\alpha}\sum_{n=0}^{\infty} e^{i(k+n)(\gamma+\varphi)-in\alpha} \quad \Big| \text{ or } \sum_{n=0}^{\infty} e^{-i(n-k+1)(\gamma+\varphi)+in\alpha}$$

So $$\int_{C'_{II}} e^{-ikr\cos\gamma}\frac{e^{ik(\gamma+\varphi)}}{e^{i(\gamma+\varphi)}-e^{i\alpha}}\,d\gamma = \begin{cases} -e^{-i\alpha}\sum_{n=0}^{\infty}\int_{C'_{I}} e^{-ikr\cos\gamma+i(k+n)(\gamma+\varphi)-in\alpha}\,d\gamma \\ \text{or} \\ \sum_{n=0}^{\infty}\int_{C'_{II}} e^{-ikr\cos\gamma-i(n-k+1)(\gamma+\varphi)+in\alpha} \end{cases}$$

Set $\alpha = 0$.

C'_{I} gives: $\displaystyle\sum_{n=0}^{\infty} -2\pi J_{k+n}(kr)\, e^{i\overset{(k+n)}{\cancel{n}}\varphi}\, e^{-\frac{\pi}{2}i(k+n)}$

C'_{II} gives $\displaystyle\sum_{n=0}^{\infty} -2\pi J_{n-k+1}(kr)\, e^{-i(n-k+1)\varphi}\, e^{-\frac{\pi}{2}i(n-k+1)}$

$$\downarrow$$

$$n-k+1 =: \quad 1-k,\ 2-k,\ 3-k,\ \cdots$$

$$= \quad |k-1|,\ |k-2|,\ |k-3|,\ \cdots \qquad \text{if } 0<k<1$$

So $\displaystyle -2\pi\sum_{n=-\infty}^{\infty} J_{|n+k|}(kr)\, e^{i(n+k)\varphi}\, e^{-i\frac{\pi}{2}|n+k|}$

$$= -2\pi e^{ik\varphi}\psi,$$

$$\psi = \sum J_{|n+k|}(kr)\, e^{in\varphi}\, (-i)^{|n+k|}$$

This ψ is the A-B solution.

Generalization : 5/31.

plane wave $\exp[i(kz + lu)]$

where $u = \bar{z}$,

$kl = m^2$ k, l are complex. $l = \bar{k}$ for actual plane wave.

Consider k, l as complex parameters.. The previous sol. can be written

$$\iint_{\Gamma} \exp[ikz] \exp[ilu] f(k, l)\, dk\, dl\, \delta(kl - m^2)$$

$$f(k,l) = k^{\alpha} l^{-\alpha} \delta(kl - m^2) / (k-k_0)(l-l_0)$$

Γ : $(\infty, 0)$ in such a way that $\text{Im}\, kz > 0$.
$\text{Im}\, lu > 0$

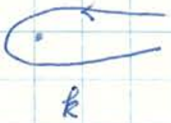
Then Γ also contains # $k \sim \frac{1}{l} \to 0$ etc

$kz \to k'|z|$ $lu \to l'|u|$

$k' = ke^{i\varphi}$ $l' = le^{-i\varphi}$ if $u = \bar{z}$

Then
Γ :

Or remove the i : write $\exp[kz \mp lu]$



The path consists of 2 pieces.

The pole factor $1/(k-k_0)(l-l_0)$ seems wrong?

We do not need two den. factors.

Use a single integral $\int dk/k-k_0$ only as $l = 1/k$

Otherwise the δ fn for complex arguments not well defined.

For N sources, the exponent is

$$\sum_n k_n (z-a_n) + \sum_n l_n (u-\bar{a}_n) = (\sum k_n) z - \sum k_n a_n + (\sum l_n) u - \sum l_n \bar{a}_n$$

and $\sum k_n \sum l_m = m^2$

Clearly we have to integrate over $2N-1$ degrees of freedom.

The pole factor:

$$\frac{1}{\sum k_n - k_0} \quad \text{or} \quad \frac{1}{\sum l_n - l_0} \quad \text{or both?}$$

Each k_n rotated by $e^{i\varphi_n}$: $z-a_n = |z-a_n| e^{i\varphi_n}$

But then $\sum k_n \to \sum k'_n e^{-i\varphi_n}$

$\{k'_n\}$ now ranges mostly in the forward cone $k'_n > 0$

At sufficiently large distances, one can make φ_n nearly equal,

so $\sum k_n \sim e^{-i\varphi} \sum k'_n$. Actually k'_n need only be in the region $\mathrm{Re}\, k'_n > 0$, so one can make all $k'_n e^{-i\varphi_n}$ have the same common phase.