Use of fields instead of potential (Pauli, Mandelstam etc)

Let $\sigma_{\mu\nu}(x)$ be ~~the~~ a Dirac string field such that

$$\partial_\mu \sigma_{\mu\nu} = \bar{j}_\nu \quad \text{given.}$$

For non-Abelian, $\quad G^a_{\mu\nu} \equiv \rho^a(x)\,\sigma_{\mu\nu}$, $\sigma_{\mu\nu} D_\mu \rho = 0$

This is possible if $\sigma_{\mu\nu}[D_\mu, D_\nu]\rho = 0 \rightarrow \rho \parallel F_{\mu\nu}\sigma_{\mu\nu}$.

So choose $\rho^a \sim \sigma\cdot F^a / |\sigma\cdot F|$

Then $\mathcal{L}_{int} = \rho^2 = |\sigma\cdot F| = [(\sigma_{\mu\nu}F_{\mu\nu})^2]^{1/2}$

$$\frac{\delta}{\delta A_\mu}\mathcal{L}_{int} = -D_\mu\left(\sigma_{\mu\nu}(x)\ \sigma\cdot F/|\sigma\cdot F|\right) = -\bar{j}_\nu\,\rho^a(x) \equiv -J_\nu^a$$

Is the min. action indep of the string configuration?

Classically, yes. The eqs motion are the same. The virial Theorem unchanged.

I don't like the $||$ sign for non-Abelian. So instead

$$\mathcal{L} = \sigma\cdot F^a \phi^a - \frac{\lambda}{2}(\phi^2 - 1)$$

$$\delta\phi : \quad \sigma\cdot F = \lambda\phi$$

$$\delta\lambda : \quad \phi^2 = 1$$

Correction : The field ρ does not exist in general.

$\rho = \sigma_{\mu\nu} \hat{F}_{\mu\nu}(x)$ does not satisfy $\sigma_{\mu\nu} D_\nu \rho = 0$.

In order to satisfy $D_1 \rho = 0$ along a sheet, we have to integrate along a path 1. The fact ~~that~~ $F_{12} = 0$ means that the ρ is ~~not~~ nonlocal & nonintegrable. Then we cannot satisfy $D_2 \rho = 0$ at the same time.

If $\rho \parallel F\cdot\sigma \rightarrow \rho \times F\cdot\sigma = 0 \rightarrow n\cdot D (\rho \times F\cdot\sigma) = 0$,

n tangent to σ. $\rightarrow \rho \times n\cdot D (F\cdot\sigma) = 0$, since $n\cdot D\rho = 0$

$\rightarrow F\cdot\sigma \times n\cdot D(F\cdot\sigma) = 0$

Let $F\cdot\sigma \rightarrow F_{12}$, $n \parallel 1 \rightarrow F_{12} \times D_1 (F\cdot\sigma)$

$\rightarrow F_{12} \times D_1 F_{12}$ since $D_1 \sigma_{12} = 0$ ~~$n\cdot D$~~

or $D_1 F_{12} \parallel F_{12}$ Call $F_{12} = F\cdot\sigma \equiv \phi$

$\rightarrow n\cdot D\phi \propto \phi \rightarrow A \times \phi = 0 \rightarrow A_1, A_2 \parallel \phi$.

$\rightarrow$ locally Abelian.

More generally, consider

$$D_\mu G_{\mu\nu} = j_\nu \rightarrow D_\nu D_\mu G_{\mu\nu} = 0 \rightarrow F_{\mu\nu} \times G_{\mu\nu} = 0$$

We may set $G_{\mu\nu} \sim \sigma_{\mu\nu}$ i.e. only one compt $\neq 0$ and so $\parallel F_{\mu\nu} \cdot \sigma_{\mu\nu}$

Then we have $G_{\mu\nu} = f(x)\, F\cdot\sigma\, \sigma_{\mu\nu}$ or $= \phi\, \sigma_{\mu\nu}$

$D_\mu G_{\mu\nu} = D_\mu \phi\, \sigma_{\mu\nu}$ is in the tangent space.

Is it possible to have $D_\mu \phi = 0$ in the tangent space?

$\sigma_{\mu\nu}\, \cancel{\partial_\mu f} = \sigma_{12}$:

$$D_1 G_{12} \rightarrow \partial_1 f\, F_{12}\cdot\sigma_{12} + f D_1(F_{12}\cdot\sigma_{12}) = 0$$

$$D_2 \quad \partial_2 f\, F_{12}\cdot\sigma_{12} + f D_2(\quad) = 0$$

These can be integrated for f. since integrability is O.K.

$$\partial_1 \ln f = -D_1 \ln|F\cdot\sigma| \qquad \cancel{\ln F\cdot\sigma}$$

$$\partial_2 \ln f = -D_2 \ln|F\cdot\sigma|$$

provided that $D_1(F\cdot\sigma) \parallel F\cdot\sigma$ or $A_1 \parallel F\cdot\sigma$

$\Rightarrow \quad A_1 \parallel A_2 \parallel F\cdot\sigma$! Again locally Abelian.

At any pt, there are four isovectors A_μ^a, so $\exists$ one relation

$$c_\mu A_\mu = 0 \rightarrow A_\lambda = 0 \text{ for some direction. } c_\lambda$$

Then we ~~may~~ (have to) choose σ to contain c_λ. ~~$\rightarrow A$~~

Construction of $\sigma_{\mu\nu}$: Given a boundary line j_ν.
Suppose j_ν does not lie in the "null" direction n. Then
$\sigma_{\mu\nu} \sim n \times j$. So $\sigma_{\mu\nu}$ is generated by n & j_ν.

If so, we may not be able to join two currents j_ν by a sheet σ.

Let's consider the following ansatz:

$$\mathcal{L} = -\frac{1}{4} F \cdot F - \frac{1}{2} F^a_{\mu\nu} \phi^a \sigma_{\mu\nu} \mp \frac{1}{4} \phi^2$$

Variables are A_μ, ϕ & the sheet variables

$$\sigma_{\mu\nu} = \iint_\sigma [y_\mu y_\nu] \, \delta^4(y-x) \, d^2\tau$$

δA_μ: $\quad -D_\nu F_{\mu\nu} \mp \underset{\substack{\shortparallel \\ (D_\nu \phi)\sigma_{\mu\nu} \mp \phi j_\mu}}{D_\nu (\phi \sigma_{\mu\nu})} = 0 \qquad j_\nu = \partial_\mu \sigma_{\mu\nu}$

$\delta \phi$: $\quad \phi = \mp F_{\mu\nu} \sigma_{\mu\nu}$

Integrability: $\quad D_\nu D_\mu (\phi \sigma_{\mu\nu}) \sim F_{\mu\nu} \times \phi \, \sigma_{\mu\nu} = 0 \quad$ O.K.

~~It's no~~ I would like to have $D_\mu \phi \, \sigma_{\mu\nu} = 0$.

This must be done in terms of a vector field V_μ:

$\mathcal{L} \sim D_\mu V_\nu \cdot \phi \, \sigma_{\mu\nu} \quad \rightarrow \quad D_\mu \phi \cdot \sigma_{\mu\nu} = 0 \quad$ But this is too much:

Even on the boundary, $j^\nu = 0$ or else $\phi = 0$ there? ~~Goo~~.

$$\cancel{(F_{\mu\nu} \times \phi)\sigma_{\mu\nu}} \quad D_\nu (D_\mu \phi \, \sigma_{\mu\nu}) \cong (F_{\mu\nu} \times \phi) \sigma_{\mu\nu} + D_\nu (\phi j_\nu) = 0$$

$$\downarrow \; \cancel{\phi D_\nu j_\nu} \; D_\nu \phi \cdot j_\nu$$

So $\quad F_{\mu\nu} \parallel \phi \quad$ <u>and</u> $\quad D_\nu \phi \cdot j^\nu = 0$

But $\delta A_\mu \rightarrow \quad V_\nu \times \phi \, \sigma_{\mu\nu} \quad$ as a new current.

$$D_\mu (V_\nu \times \phi) \sigma_{\mu\nu} = D_\mu V_\nu \times \phi \, \sigma_{\mu\nu} \rightarrow 0$$

Next try a piece $\mathcal{L} \sim D_\mu\phi \times D_\nu\phi \cdot \phi\, \sigma_{\mu\nu}$

$$\delta A_\mu \rightarrow \quad \cancel{-2 D_\mu (D_\nu\phi \times \phi\, \sigma_{\mu\nu})} \qquad D_\mu = \partial_\mu + A_\mu \times \quad \text{so}$$

$$D_\mu\phi \times D_\nu\phi \cdot \phi = \partial_\mu\phi \times D_\nu\phi \cdot \phi + [(A_\mu \times \phi) \times D_\nu\phi] \cdot \phi$$

$$\downarrow$$

$$(A_\mu \times \phi) \cdot (D_\nu\phi \times \phi)$$

$$\downarrow$$

$$A_\mu \cdot [\phi \times (D_\nu\phi \times \phi)]$$

$$\rightarrow \quad 2\,[\phi \times (D_\nu\phi \times \phi)]\, \sigma_{\mu\nu}$$

$$= 2\, D_\nu\phi\, \sigma_{\mu\nu} - \phi \cdot D_\nu\phi\, \phi\, \sigma_{\mu\nu}$$

$$= 2 D_\nu\phi\, \sigma_{\mu\nu} \quad \text{if} \quad \phi^2 = \text{const.} = 1.$$

Therefore choose $\mathcal{L} = +\frac{1}{2}(D_\mu\phi \times D_\nu\phi) \cdot \phi\, \sigma_{\mu\nu}$ //

Then $$D_\nu F_{\mu\nu} + \underbrace{D_\nu(\phi\, \sigma_{\mu\nu}) - (D_\nu\phi)\sigma_{\mu\nu}}_{\phi j_\mu\ !} = 0$$

$$\delta\phi : \quad -\frac{1}{2} F_{\mu\nu}\sigma_{\mu\nu} - \frac{1}{2}\phi + \frac{1}{2} D_\mu\phi \times D_\nu\phi\, \sigma_{\mu\nu} - D_\mu(D_\nu\phi \times \phi\, \sigma_{\mu\nu})$$

$$= 0$$

$$D_\mu(D_\nu\phi \times \phi\, \sigma_{\mu\nu}) = (D_\mu D_\nu\phi) \times \phi\, \sigma_{\mu\nu} + (D_\nu\phi \times D_\mu\phi)\, \sigma_{\mu\nu}$$

$$+ (D_\nu\phi \times \phi)\, j^\nu$$

$$= (F_{\mu\nu} \times \phi) \times \phi\, \sigma_{\mu\nu} + (D_\nu\phi \times D_\mu\phi)\, \sigma_{\mu\nu} + (D_\nu\phi \times \phi)\, j_\nu$$

$$\parallel$$

$$\phi\, F_{\mu\nu} \cdot \phi\, \sigma_{\mu\nu} - F_{\mu\nu}\sigma_{\mu\nu}$$

$$\rightarrow \quad +\frac{1}{2} F_{\mu\nu} \cdot \sigma_{\mu\nu} - \frac{1}{2}\phi - \phi\, F_{\mu\nu} \cdot \phi\, \sigma_{\mu\nu} + \frac{3}{2} D_\mu\phi \times D_\nu\phi\, \sigma_{\mu\nu} - (D_\nu\phi \times \phi)\, j_\nu$$

$$\parallel$$

$$0.$$

$$\rightarrow \quad \cancel{F_{\mu\nu}\sigma_{\mu\nu} + D_\mu\phi \times D_\nu\phi \cdot \sigma_{\mu\nu} = 0\ \parallel \phi}$$

$$\cancel{F \cdot \sigma = \frac{1}{2}}. \qquad F_{\mu\nu}\sigma_{\mu\nu} \parallel \phi$$

By contacting with ϕ:

$$-\cancel{\phi}\,\tfrac{1}{2}\,\phi\cdot F_{\mu\nu}\sigma_{\mu\nu} - \tfrac{1}{2} + \tfrac{3}{2}\,\phi\cdot D_\mu\phi\times D_\nu\phi\,\sigma_{\mu\nu} = 0$$

Rather, with a Lagrange multiplier $\lambda(\phi^2-1)$ in $\mathcal{L}$,

$\frac{1}{2}$ should be replaced by $\lambda/2$, a field.

So there is no constraint except that $\phi \parallel F_{\mu\nu}\cdot\sigma_{\mu\nu}$ Good!

The $\mathcal{L}$ amounts to taking

$$(F_{\mu\nu} \mp D_\mu\phi\times D_\nu\phi)\cdot\phi\,\sigma_{\mu\nu}$$

$D_\mu\phi\times D_\nu\phi$ contains $(A_\mu\times\phi)\times(A_\nu\times\phi) = -(A_\mu\times\phi)\cdot A_\nu\,\phi$
$= (A_\mu\times A_\nu)\cdot\phi\;\phi$

$F_{\mu\nu} = (\underbrace{\partial_\mu + A_\mu\times)(\partial_\nu}+A_\nu\times)$

$$A_\mu\times(A_\nu\times X) - A_\nu\times(A_\mu\times X)$$
$$= \cancel{A_\mu\times}$$
$$= \underbrace{A_\mu\cdot X\,A_\nu} - \underbrace{A_\mu\cdot A_\nu}\,X$$
$$= (A_\mu\times A_\nu)\times X$$

$$(A_\mu\times\phi)\times\partial_\nu\phi = A_\mu\cdot\partial_\nu\phi\;\phi - A_\mu\;\phi\cdot\partial_\nu\phi$$
$$= A_\mu\cdot\partial_\nu\phi\;\phi$$

So $(F_{\mu\nu} \mp D_\mu\phi\times D_\nu\phi)\cdot\phi \rightarrow F_{\mu\nu}\cdot\phi \mp (A_\mu\times A_\nu)\cdot\phi \mp \underbrace{A_\mu}\cdot\partial_\nu\phi - \partial_\mu\phi\times\partial_\nu\phi\cdot\phi$

$$\downarrow$$

$$\partial_\nu(\underbrace{A_\mu}\cdot\phi) + \partial_\mu \underbrace{A_\nu}\cdot\phi$$

$$\rightarrow \cancel{2F_{\mu\nu}\cdot\phi\,\phi} + \partial_\mu(\underbrace{A_\nu}\cdot\phi)\;\cancel{\phi} - \partial_\mu\phi\times\partial_\nu\phi\cdot\phi$$

If $\phi^2 \neq 1$, additional term:
~~Something is wrong.~~

$$(1-\phi^2)[F_{\mu\nu}\cdot\phi - \partial_\mu(\underbrace{A_\nu}\cdot\phi)]$$

N.B. The combination $(F_{\mu\nu} - D_\mu\phi \times D_\nu\phi)\cdot\phi$

must be the 't Hooft field which can also be written

$$\underbrace{\partial_\mu(\phi\cdot A_\nu)} - \partial_\mu\phi\times\partial_\nu\phi\cdot\phi$$

My form, when $D_\mu\phi \to 0$, is

$$F_{\mu\nu} = \partial_\mu\phi\times\partial_\nu\phi + \phi\, \overset{f}{a}_{\mu\nu} \quad\to\quad \phi\cdot F_{\mu\nu} - \phi\cdot\partial_\mu\phi\times\partial_\nu\phi = f_{\mu\nu}$$

The relation is not so transparent.

At any rate, we have the total $\mathcal{L}$

$$\mathcal{L} = -\frac{1}{4}F\cdot F - \frac{1}{2}G\cdot(F-H) \qquad , \qquad H_{\mu\nu} = D_\mu\phi\times D_\nu\phi$$
$$- \frac{\lambda}{4}\phi^2 \qquad\qquad G_{\mu\nu} = \phi\,\sigma_{\mu\nu}$$

Variation of y_μ:

$$\frac{\delta}{\delta y_\lambda}\sigma_{\mu\nu} =: -\frac{\partial}{\partial\tau}\left(y_\nu'\,\delta(y-x)\right)\delta_{\mu\lambda} + \frac{\partial}{\partial\tau}\left(y_\mu'\,\delta(y-x)\right)\delta_{\nu\lambda} + (\tau\leftrightarrow\sigma)$$
$$+ (\dot y_\mu y_\nu' - \dot y_\nu y_\mu')\frac{\partial}{\partial y_\lambda}\delta(y-x)$$

$$= -[y_\nu'\dot y_\rho]\frac{\partial}{\partial y_\rho}\delta(\cancel{y-x})\,\delta_{\mu\lambda} + [y_\mu'\dot y_\rho]\frac{\partial}{\partial y_\rho}\delta(y-x)\,\delta_{\nu\lambda}$$
$$+ (\dot y_\mu y_\nu' - \dot y_\nu y_\mu')\frac{\partial}{\partial y_\lambda}\delta(y-x)$$

~~$[y\rho$~~

$$\frac{\delta}{\delta y_\lambda}: X_{\mu\nu}\sigma_{\mu\nu} \to -X_{\lambda\nu}[y_\nu'\dot y_\rho]\frac{\partial}{\partial y_\rho}\delta(y-x) + X_{\mu\lambda}[y_\mu'\dot y]_\rho\frac{\partial}{\partial y_\rho}\delta(y-x)$$
$$+ X_{\mu\nu}[y_\mu y_\nu]\frac{\partial}{\partial y_\lambda}\delta(y-x)$$

$$= X_{\lambda\nu}[y_\nu'\dot y_\rho]\frac{\partial}{\partial x_\rho}\delta(y-x) - X_{\mu\lambda}[y_\mu'\dot y_\rho]\frac{\partial}{\partial x_\rho}\delta(y-x)$$
$$- X_{\mu\nu}[y_\mu y_\nu]\frac{\partial}{\partial x_\lambda}\delta(y-x)$$

$$\to -[y_\rho y_\mu] \frac{\partial}{\partial x_\rho} X_{\lambda\mu} \mp [y_\rho y_\mu] \frac{\partial}{\partial x_\rho} X_{\lambda\mu} + [y_\mu y_\nu] \frac{\partial}{\partial x_\lambda} X_{\mu\nu} \times \delta(y-x)$$

$$= [y_\mu y_\nu] \sum_{cyc} \frac{\partial}{\partial x_\lambda} X_{\mu\nu} \; \delta(y-x)$$

Here $X_{\mu\nu} = (F_{\mu\nu} - D_\mu\phi \times D_\nu\phi)\cdot\phi$

$$\sum \partial_\lambda X_{\mu\nu} = \sum D_\lambda\phi\cdot(F_{\mu\nu} - D_\mu\phi\times D_\nu\phi)$$

$$- \sum D_\lambda D_\mu\phi \times D_\nu\phi\cdot\phi - \sum D_\mu\phi\times D_\lambda D_\nu\phi\cdot\phi$$

$$-\tfrac{1}{2}\sum[(F_{\lambda\mu}\times\phi)\times D_\nu\phi]\cdot\phi - \tfrac{1}{2}\sum[D_\mu\phi\times(F_{\lambda\nu}\times\phi)]\cdot\phi$$

$$= -\sum D_\lambda\phi\cdot D_\mu\phi\times D_\nu\phi$$

$$- \sum \underset{\overset{\parallel}{0}}{\phi\cdot D_\nu\phi} \; F_{\lambda\mu}\cdot\phi \qquad \to 0$$

0 if $\phi^2 = 1$.

$$= \sum D_\lambda\phi\cdot D_\mu\phi\times D_\nu\phi$$

also $= \sum_{cycl.} \partial_\lambda\phi\cdot\partial_\mu\phi\times\partial_\nu\phi = 0$ unless $\vec{\phi}$ is singular.

This is the ~~mon~~ magnetic current $k_\rho = \frac{1}{2}\varepsilon_{\rho\lambda\mu\nu}\,\partial_\lambda\phi\,\partial_\mu\phi\,\partial_\nu\phi$!

so $\sum \partial_\lambda X_{\mu\nu}\,\sigma_{\mu\nu} = -\tilde{\sigma}_{\lambda\rho} k_\rho = -\frac{1}{2}\varepsilon_{\lambda\rho\alpha\beta}\,\sigma_{\alpha\beta}$

$$\times \tfrac{1}{2}\varepsilon_{\rho\tau\mu\nu}\,\partial_\tau\phi\,\partial_\mu\phi\,\partial_\nu\phi$$

Given a random distribution of singularities of ϕ, how to ~~show~~ derive the area law?

Consider the correlation:-

$$Z = \exp[i \int \mathcal{L} \, d^4x]$$

$$\frac{\delta}{\delta y'_\sigma} \frac{\delta Z}{\delta y_\lambda} \Rightarrow \quad - \langle \tilde{\sigma}_{\lambda\rho} k_\rho \times \tilde{\sigma}'_{\sigma\tau} k'_\tau \rangle$$

$$\langle k_\rho k'_\sigma \rangle \to C \delta_{\rho\sigma} \quad \text{if the two points coincide.}$$

$$\Rightarrow \; - C \tilde{\sigma}_{\lambda\rho} \tilde{\sigma}_{\sigma\rho} Z$$

Probably this is not the correct treatment.

Given a current k_ρ, ~~δZ~~ $\delta Z / \delta y_\lambda \neq 0$, so the action is not at a minimum. By changing y_λ, it will reach a lower action configuration. So the min. will be reached, on the ~~average~~ average, by moving by a ~~distance of k~~ "mean free path" or average distance between monopoles.

Monopole current $\quad k_{\mu} = \varepsilon_{\mu\nu\lambda\rho}\, \partial_\nu \phi^a \partial_\lambda \phi^b \partial_\rho \phi^c\, \varepsilon^{abc}$

$= \partial_\lambda n_{\lambda\mu}$?

$n_{\lambda\mu} = \varepsilon_{\lambda\mu\nu\rho}\, \partial_\nu \phi^a \partial_\rho \phi^b \phi^c\, \varepsilon^{abc}$? yes.

and this is almost conserved : $\partial_\lambda n_{\lambda\mu} \approx 0$.

Generalization of $D_\mu \phi = 0$:

$$(D_\mu + V_\mu \times)\,\phi = 0$$

$$(D_\nu + V_\nu \times)\,\phi = 0 \quad \rightarrow \quad (\underbrace{[D_\mu D_\nu]}_{F_{\mu\nu}} + D_\mu V_\nu + V_\mu \times V_\nu)\cdot \phi = 0$$

Also $\quad V_\mu = -\phi \times D_\mu \phi + \phi a_\mu$

Check $\quad \phi \times D_\mu \phi + \underbrace{\phi \times (V_\mu \times \phi)}_{V_\mu - \phi\, V_\mu\cdot\phi} = 0$

$$D_\mu V_\nu = D_\mu(-\phi \times D_\nu \phi) + D_\mu(\phi a_\nu)$$

$$= -2 D_\mu \phi \times D_\nu \phi - \phi \times (D_\mu D_\nu \phi) + D_\mu(\phi a_\nu)$$

$$V_\mu \times V_\nu = (-\phi \times D_\mu \phi + \phi a_\mu) \times (-\phi \times D_\nu \phi + \phi a_\nu)$$

$$= (\phi \times D_\mu \phi) \times (\phi \times D_\nu \phi) - (\phi \times D_\mu \phi) \times \phi\, a_\nu - \phi a_\mu \times (\phi \times D_\nu \phi)$$

$$= +(\phi \times D_\mu \phi)\cdot D_\nu \phi\, \phi - D_\mu \phi\, a_\nu + \phi\, \underbrace{D_\mu \phi \cdot \phi}_{0}\, a_\nu$$

$$D_\mu V_\nu + V_\mu \times V_\nu = -D_\mu \phi \times D_\nu \phi \cdot \phi\, \phi + \phi\, \partial_\mu a_\nu - \underbrace{\phi \times (F_{\mu\nu} \times \phi)}_{-F_{\mu\nu} + \phi\, \phi\cdot F_{\mu\nu}}$$

So $\quad \cancel{\Phi\cdot}\, F_{\mu\nu} + D_\mu V_\nu + V_\mu \times V_\nu$

$$= \phi\, \phi\cdot F_{\mu\nu} - \phi\, D_\mu \phi \times D_\nu \phi \cdot \phi + \phi\, \partial_\mu a_\nu$$

$$= \phi\, [\phi\cdot F_{\mu\nu} - D_\mu \phi \times D_\nu \phi \cdot \phi + \partial_\mu a_\nu]$$ which is $\parallel \phi$.

Projection onto an arbitrary ϕ of this combination is always integrable.

A New look:

Conditions for integrable phase factor

Let $\tilde{D}_\mu \equiv \partial_\mu + A_\mu \times + V_\mu \times = D_\mu + V_\mu \times$ or $= \partial_\mu + \tilde{A}_\mu$

and $U = P\exp[i \int \tilde{A}_\mu dx^\mu]$ $\qquad \tilde{A}_\mu = A_\mu + V_\mu$

$$\frac{\delta}{\delta\sigma_{\mu t}} U = i U' \tilde{F}_{\mu t} U''$$

This $\tilde{F}_{\mu\nu}$ can be displaced if $\tilde{D}_t \tilde{F}_{\mu t} = 0$:

$$\frac{\delta U(x\, y)}{\delta\sigma_\mu t(z)} = i \tilde{F}_{\mu t}(x)\, U(x,y).$$

$$\rightarrow \quad U_T(\overrightarrow{x\,y}) = \exp\left[i L \int^T \tilde{F}_{03}(t)\, dt\right] U_0(\overset{x\,y}{})$$

If ~~further~~ $\cancel{\tilde{D}_0 \tilde{F}_{03} = 0}$

$$\cancel{\rightarrow \quad = \exp[i LT \tilde{F}_{03}(x,T)]\, U_0(x,y)}$$

$$\cancel{= U_0(x,y) \exp[i LT \tilde{F}_0(y,T)]}$$

This assumes a flat surface.

So define $\tilde{D}_\mu = D_\mu + V\times$; or $\tilde{A}_\mu = A_\mu + V_\mu$

$$\tilde{F}_{\mu\nu} = [\tilde{D}_\mu, \tilde{D}_\nu] = \cancel{\phi}\, F_{\mu\nu} + \underbrace{D_\mu V_\nu} + V_\mu \times V_\nu$$

$(\mu,\nu)=(0,3)$ only

$$\tilde{D}_0 \tilde{F}_{03} = \tilde{D}_3 \tilde{F}_{30} = 0 \quad \to \text{call } \tilde{F}_{03} \overset{=\kappa^2}{=} \phi, \quad \phi^2 = \cancel{\text{const}}\ 1$$

$$\tilde{A}_\mu = -\phi\times\partial_\mu\phi + \phi\tilde{a}_\mu \quad \text{or} \quad V_\mu = -\phi\times D_\mu\phi + \phi\, a_\mu$$

$$\tilde{a} = a + A\cdot\phi \quad , \quad a: \text{ gauge covariant}$$

$$\tilde{F}_{\mu\nu} = F_{\mu\nu} + \cancel{\quad} \underbrace{D_\mu V_\nu} + V_\mu\times V_\nu$$

$$\tilde{F}_{\mu\nu} = -\partial_\mu\phi\times\partial_\nu\phi + \phi\underbrace{\partial_\mu\tilde{a}_\nu}$$

$$\underbrace{D_\mu V_\nu} + V_\mu\times V_\nu = -F_{\mu\nu} + \phi\,\phi\cdot F_{\mu\nu} - \left(D_\mu\phi\times D_\nu\phi\cdot\phi - \underbrace{\partial_\mu a_\nu}\right)\phi$$

$$\tilde{F}_{\mu\nu} = \phi\,[\phi\cdot F_{\mu\nu} - D_\mu\phi\times D_\nu\phi + \underbrace{\partial_\mu a_\nu}]$$

Field eq. $\to$ $\quad \phi\cdot F_{\mu\nu} - D_\mu\phi\times D_\nu\phi + \underbrace{\partial_\mu a_\nu} = \cancel{\phi}\text{ const} = \kappa^2$

to determine a_μ for given $F_{\mu\nu}$ & ϕ.

What is the effective $\mathcal{L}$?

Extension to $SU(3)$ is not trivial:

One way to proceed is to take an octet ϕ^a

and complete $\underbrace{\partial_\mu (A_\nu \cdot \phi)}$ by an additional term.

$$= \underbrace{\partial_\mu A_\nu^a} \cdot \phi^a + \underbrace{\partial_\mu \phi^a A_\nu^a}$$

$$\downarrow$$

$$F_{\mu\nu}^a \cdot \phi^a =: f^{abc} A_\mu^b A_\nu^c \phi^a$$

so $\underbrace{\partial_\mu \phi^a A_\nu^a} - f^{abc} A_\mu^b A_\nu^c \phi^a + X_{\mu\nu}$ X indep. of A.

$$(D_\mu \phi)^a = \partial_\mu \phi^a + f^{abc} A_\mu^b \phi^c$$

$$(D_\mu \phi)^a \cdot A_\nu^a = \partial_\mu \phi^a \cdot A_\nu^a + f^{abc} A_\mu^b \phi^c A_\nu^a$$

$$\qquad\qquad\qquad\qquad = - f^{abc} A_\mu^b A_\nu^c \phi^a$$

~~Thus ... $\partial_\mu \phi^a$ ... $X_{\mu\nu}$~~

Try $(D_\mu \phi)^a (D_\nu \phi)^b \chi^c f^{abc}$

contains $\underbrace{\partial_\mu \phi^a \quad A_\nu^l} \phi^m f^{blm} \chi^c f^{abc}$

$$+ f^{aij} A_\mu^i \phi^j f^{blm} A_\nu^l \phi^m \chi^c f^{abc}$$

$$\to \underbrace{\partial_\mu \phi^a A_\nu^l} \quad \underbrace{\phi^m \chi^c f^{blm} f^{abc}} + \cdots$$

should be $= \delta^{al} \phi \cdot \chi$ (1)

and $A_\mu^i A_\nu^l \phi^j \phi^m \chi^c \cdot f^{aij} f^{blm} f^{abc}$

should be $= - f^{ilk} \phi^k A_\mu^i A_\nu^l$

or $\phi^j \phi^m \chi^c f^{aij} f^{blm} f^{abc} = - f^{ilk} \phi^k$ (2)

$$f^{abc} f^{blm} = ?$$

If ① is satisfied, Then ② $= \phi^j f^{aij} \delta^{al} = \phi^j f^{lij}$ o.K.

So we need only ①

But contract ① with χ^a : $\quad 0 = \chi^l$!

Thus it cannot be $\sim \delta^{al}$.

Rather, we can add a term which lead to $\partial_i \phi \cdot \phi \to 0$.

i.e. $\sim \phi^a \chi^l$

In the case of $SU(2)$:

$$\varepsilon^{abc} \varepsilon^{blm} = - \varepsilon^{bac} \varepsilon^{blm} = - \delta^{al} \delta^{mc} + \delta^{am} \delta^{lc}$$

o.K.

For $SU(3)$ this is not so!

$$f^{bac} f^{blm} \equiv T^{(\widehat{ac}, \widehat{lm})}$$

~~It is the [al] [mc] part of this tensor?~~

But if $\phi \approx \chi$, we are taking the symmetric part (mc) of T.

Let ϕ be $\alpha \phi^3 + \beta \phi^8$ Then only terms $l = a$ contribute. So we should choose ϕ & χ to be commuting.

A trivial solution:

Let u, u^+ be isospinor (or $SU(3)$ $\underline{3}$)

$F_{\mu\nu} = \partial_\mu A_\nu - i[A_\mu A_\nu]$
$D_\mu = \partial_\mu - iA_\mu$

Write the current term as $\frac{i}{2} u^+ \overleftrightarrow{D_\mu} u\, j^\mu$

$= u^+ A_\mu u\, j^\mu + \frac{i}{2} u^+ \overleftrightarrow{\partial_\mu} u\, j^\mu$

So the source is simply $u^+ \frac{\tau^i}{2} u\, j^\mu$ or $\phi^i = u^+ \frac{\tau_i}{2} u$!

This is certainly covariant. There is a $U(1)$ -like contribution

$A^0_\mu = \frac{i}{2} u^+ \overleftrightarrow{\partial_\mu} u$!

We can also write it as a sheet:

$$\frac{1}{2}\iint \left(u^+ \dot{F}_{\mu\nu} u + i \underbrace{D_\mu u^+ D_\nu u}\right) \sigma_{\mu\nu} = \frac{1}{2}\iint \left[\partial_\mu \underbrace{(u^+ A_\nu u)} + i \underbrace{\partial_\mu u^+ \partial_\nu u}_{\underbrace{\partial_\mu A^0_\nu}} \right] \sigma_{\mu\nu}$$

In this way, no ~~intega~~ shape dependence enters. Alas!

Wait! A^0_μ ~~has~~ can have a string singularity.

How can u become singular? Assume the $SU(2)$ (or $SU(3)$) part is regular.

$$u \sim \begin{pmatrix} a \\ b e^{i\phi n} \end{pmatrix} \qquad \phi \sim \text{azimuthal angle.}$$

$u^\dagger \tau^i u$ nonsingular if $ab = 0$. So $a = 0$ if $b \neq 0$.
$\rightarrow b = 1$.

Then $-i u^\dagger \partial_\phi u = \frac{n}{\phi}$ is singular.

$$\oint -i u^\dagger \partial_\phi u \cdot d\phi = 2\pi n$$

Similarly for $SU(3)$: $\quad u \sim \begin{pmatrix} a \\ b e^{i\phi n} \\ c e^{i\phi m} \end{pmatrix}$

$\rightarrow a = 0, \quad n = m; \quad b, c \neq 0$
or $a = 0, \quad b = 0, \quad c = 1$ or $(b = 1, \; c = 0)$

$$-i u^\dagger \partial_\phi u = (b^2 + c^2) n = n.$$

Suppose, however, that u itself is not one-valued, e.g.

$$u \sim \begin{pmatrix} a e^{-i\phi/2} \\ b e^{+i\phi/2} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\theta/2 \; e^{i\phi/2} \end{pmatrix}$$

equivalent to $\begin{pmatrix} a \\ b e^{i\phi} \end{pmatrix}$ for $u^\dagger \tau^i u$.

Then $-i u^\dagger \partial_\phi u = -\frac{1}{2}(a^2 - b^2) \rightarrow \frac{1}{2}$ as $a \rightarrow 0$.

$$\frac{i}{2} u^+ \overleftrightarrow{\partial_\mu} u = \frac{i}{2} u_\alpha^+ \overleftrightarrow{\partial_\mu} u_\alpha \qquad u_\alpha = \sqrt{\rho_\alpha}\, e^{-i\theta_\alpha}$$

$$\rightarrow \quad = \sum_\alpha \rho_\alpha \partial_\mu \theta_\alpha \qquad \sum \rho_\alpha = 1.$$

This is like a pure gauge, or almost.

Suppose u is replaced by a set of u's : $\{u_n\}$

and demand only that $\sum_n u_n^+ u_n = 1$

Then each u_n need not be normalized to unity.

If one of them has a singularity, it will contribute

$$\int i u_n^+ \partial_\phi u_n = |u_n|^2 2\pi$$

Thus the action is surface-dependent!

Examples. 1. $u \to q \qquad u^+ \partial_\mu u \to \bar{q} \partial_\mu q$

$\bar{q} q$ not ≥ 0.

2. u is a flavor doublet.

Remarks on the Stokes' formula.

① $\frac{i}{2}\int u^+ \overleftrightarrow{D} u \, j^\mu$ is not equal to

② $\frac{1}{2}\iint (u^+ F_{\mu\nu} u + i \underbrace{D_\mu u^+ D_\nu}_{} u)\, \sigma_{\mu\nu} = \frac{1}{2}\iint [\underbrace{\partial_\mu (u^+ A_\nu u)}_{} + i \underbrace{\partial_\mu u^+ \partial_\nu}_{} u]\, \sigma_{\mu\nu}$

because $\underbrace{\partial_\mu \partial_\nu}_{} u \neq 0$.

$$② = ① - \frac{i}{2}\iint u^+ \underbrace{\partial_\mu \partial_\nu}_{} u \, \sigma_{\mu\nu}$$

$$\text{or } ① + \frac{i}{2}\iint \underbrace{\partial_\mu \partial_\nu}_{} u^+ u \, \sigma_{\mu\nu}$$

∂_μ behaves like a covariant derivative : $[\partial_\mu, \partial_\nu]$

$\approx (\exp[i\oint A\cdot dx] - 1)\,/\,\text{area}$

$\to i 2\pi n \delta^2(x)$

In other words, $\frac{i}{2}\iint \quad = \sum_i 2\pi n_i$

This looks like the formula of residues.

If $u^+u \neq 1$, we have $\sum_i 2\pi n_i \rho_i$

It measures the amount of vortices on the surface.

Winding numbers

Consider an area & a line integral

$$I = \iint d\sigma^{(x)}_{\mu\nu} \int dy_\lambda \, X_\rho \, \varepsilon_{\mu\nu\lambda\rho} \qquad X_\rho = \frac{\partial}{\partial x_\rho} f((x-y)^2)$$

$$= \iint \underbrace{\dot{x}_\mu x'_\nu} \, d\tau d\sigma \int \mathring{y}_\lambda ds \, X_\rho \, \varepsilon_{\mu\nu\lambda\rho}$$

with $x_\mu(\tau,\sigma)$, $y_\lambda(s)$ given.

Variation of x : $\cancel{\delta y_\alpha}$ δx_α

$$\delta I / \delta x_\alpha(\tau,\sigma) = \underbrace{\dot{x}_\mu x'_\nu} \frac{\partial}{\partial x_\alpha} \int X_\rho \, \varepsilon_{\mu\nu\lambda\rho} \, \mathring{y}_\lambda \, ds$$

$$- \frac{\partial}{\partial \tau} x'_\nu X_\rho \cdots \quad \varepsilon_{\alpha\nu\lambda\rho} \cdots$$

$$+ \frac{\partial}{\partial \tau} x'_\mu X_\rho \cdots \quad \varepsilon_{\mu\alpha\lambda\rho} \cdots$$

$$+ \frac{\partial}{\partial \sigma} \dot{x}_\nu X_\rho \cdots \quad \varepsilon_{\alpha\nu\lambda\rho} \cdots$$

$$- \frac{\partial}{\partial \sigma} \dot{x}_\mu X_\rho \cdots \quad \varepsilon_{\mu\alpha\lambda\rho} \cdots$$

$$= \underbrace{\dot{x}_\mu x'_\nu} \underset{\substack{\downarrow \\ \text{cycl}(\alpha\mu\nu)}}{\sum} \frac{\partial}{\partial x_\alpha} X_\rho \, \varepsilon_{\mu\nu\lambda\rho} \, \mathring{y}_\lambda =:$$

Fix $\mu\nu\alpha = 123$ $\qquad \alpha = 3.$

Call $\underbrace{\dot{x}_\mu x'_\nu} \equiv \Sigma_{\mu\nu}$

$$\delta I/\delta x_\alpha = \Sigma_{\mu\nu} \sum_{(\alpha\mu\nu)} \partial_\alpha X_\rho \, \varepsilon_{\mu\nu\lambda\rho} \, \dot{y}_{\rho\lambda} \quad \cancel{2\Sigma^*_{\lambda\rho} \sum \partial_\lambda}$$

If $\alpha = 1$:

$$2\Sigma_{23}\left(\partial_1 \underbrace{X_4 \dot{y}_1} + \partial_2 X_4 \dot{y}_2 + \partial_3 X_4 \dot{y}_3\right)$$
$$+ 2\Sigma_{34}\left(\partial_1 \underbrace{X_2 \dot{y}_1} + \cdots \qquad\right)$$
$$+ 2\Sigma_{42}\left(\partial_1 \underbrace{X_3 \dot{y}_1} + \cdots \qquad\right)$$

$$= 2\Sigma^*_{14}\left(\quad \cdots \quad \right)$$
$$+ 2\Sigma^*_{12}\left(\qquad \right)$$
$$+ 2\Sigma_{13}\left(\qquad \right)$$

$$= 2\Sigma^*_{1\beta} \, \partial_\alpha \underbrace{X_\beta \dot{y}_\alpha}$$

or in general

$$\delta I/\delta x_\alpha = 2\Sigma^*_{\alpha\beta} \int \frac{\partial}{\partial x_\alpha} \underbrace{X_\beta \dot{y}_\alpha} \, ds$$

Let $X_\beta = \dfrac{\partial}{\partial x_\beta} \dfrac{1}{X^2}$

$$\frac{\partial}{\partial x_\alpha} \underbrace{X_\beta \dot{y}_\alpha} = \int \left[\frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} \frac{1}{X^2} \dot{y}_\alpha - \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\alpha} \frac{1}{X^2} \dot{y}_\beta \right]$$

$\downarrow$

$\sim \delta^4(x-y)$

$$1\text{st term} = -\frac{\partial}{\partial x_\beta} \int \frac{d}{ds} \frac{1}{(x-y(s))^2} \to 0$$

so $$\delta I/\delta x_\alpha = 2\Sigma^*_{\alpha\beta}(x) \int \delta^4(x-y) \, dy_\beta = 2\Sigma^*_{\alpha\beta}(x) \, j_\beta(x)$$

Similarly for a variation of y_α:

$$\delta I / \delta y_\alpha = \int d\tau d\sigma \Sigma_{\mu\nu}(x) \left[\varepsilon_{\mu\nu\lambda\rho} \left(\frac{\partial}{\partial y_\alpha} X_\rho \right) \mathring{y}_\lambda - \varepsilon_{\mu\nu\alpha\rho} \mathring{X}_\rho \right]$$

$$\mathring{X}_\rho = \frac{\partial}{\partial y_\rho} X_\rho \mathring{y}_\rho$$

$$\cancel{\frac{\partial}{\partial y_\alpha} X_\rho = -\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\rho}\frac{1}{\partial X^2} \to 0}$$

~~but we performed partial integration~~

$$\cancel{X_\rho \delta}$$

$$= \int d\tau d\sigma \Sigma_{\mu\nu} \; \varepsilon_{\mu\nu\lambda\rho} \underbrace{\frac{\partial}{\partial y_\alpha}} X_\rho \mathring{y}_\lambda$$

$$= -2 \int \Sigma^*_{\lambda\rho} \underbrace{\frac{\partial}{\partial x_\alpha}} X_\rho \mathring{y}_\lambda \quad \cancel{= 2\int\frac{\partial}{\partial}}$$

~~Choose $\alpha = 1$~~ $\quad \varepsilon_{\mu\nu\lambda\rho} \partial_\alpha X_\rho \mathring{y}_\lambda$

$$= \varepsilon_{\mu\nu\lambda\rho} \partial_\alpha X_\rho \mathring{y}_\lambda \; - \; \varepsilon_{\mu\nu\alpha\rho} \partial_\lambda X_\rho \mathring{y}_\lambda$$

Also $\sum_{(\lambda\rho\alpha)} \varepsilon_{\mu\nu\lambda\rho} \partial_\alpha X_\rho \mathring{y}_\lambda = \varepsilon_{\mu\nu\lambda\rho} \partial_\alpha X_\rho \mathring{y}_\lambda + \varepsilon_{\mu\nu\rho\alpha} \partial_\lambda X_\rho \mathring{y}_\lambda + \varepsilon_{\mu\nu\alpha\lambda} \partial_\rho X_\rho \mathring{y}_\lambda$

$$= \varepsilon_{\mu\nu\lambda\rho} \partial_\alpha X_\rho \mathring{y}_\lambda + \varepsilon_{\mu\nu\alpha\lambda} (\partial_\rho X_\rho) \mathring{y}_\lambda$$

Now what is $\sum_{(\lambda\rho\alpha)} \varepsilon_{\mu\nu(\lambda\rho} \partial_{\alpha)} X_\rho \mathring{y}_\lambda$?

$$\cancel{\varepsilon_{\mu\nu\lambda\rho} \partial_\alpha \varepsilon_{\lambda\rho\alpha\beta} = 2(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})}$$

$$\cancel{\to \times \Sigma_{\mu\nu} \partial_\alpha X_\rho \mathring{y}_\lambda \to 2 \Sigma_{\alpha\beta} \partial_\alpha}$$

$$\sum_{(\lambda\rho\alpha)} \varepsilon_{\mu\nu\lambda\rho}\, \underbrace{\partial_\alpha X_\rho}\, \mathring{y}_\lambda$$

$(\lambda\rho\alpha) = 123:$ $\quad \varepsilon_{\mu\nu 12}\, \partial_3 X_2 \underbrace{\mathring{y}_1} + \varepsilon_{\mu\nu 31}\, \partial_2 X_{\underbrace{2}} \mathring{y}_1 + \varepsilon_{\mu\nu 23}\, \partial_1 X_2 \underbrace{\mathring{y}_1}$

$\varepsilon_{\mu\nu 23}\, \partial_2 X_3 \underbrace{y_1}$

This must $\to 0$ because $\partial_\alpha X_\rho = \partial_\alpha \partial_\rho f$ is symmetric in $(\alpha\rho)$?