Dec 14. Instantons revisited.

London ansatz (L.A.)

$$A_\mu = \lambda j_\mu \qquad j_\mu = -\frac{i}{2}\sum_{n=1}^{N} U_n^+ \overleftrightarrow{D_\mu} U_n$$

Extend this to $J_\mu = 0$ where

$$J_\mu = -\frac{i}{2}\sum_{n=0}^{N} U_n^+ \overleftrightarrow{D_\mu} U_n \quad , \text{ and}$$

$U_0 =$ const. (spontaneous breakdown!)

This will reproduce the L.A.

Now we want to know if U_n individually will satisfy an eq. motion.

Let $j_\mu = -\frac{i}{2} u^+ \overleftrightarrow{D_\mu} U = -i f u^+ D_\mu u \quad , \quad U = \sqrt{f}\, u,$
$u^+ u = 1$

so $\quad D_\mu u = \frac{i}{f} u j_\mu \quad , \quad D_\mu j_\mu = 0$

$$\Rightarrow \quad D_\mu D_\mu u = i D_\mu\left(\frac{1}{f}u\right) j_\mu = -i \frac{\partial_\mu f}{f^2} u j_\mu + i \frac{1}{f} D_\mu u j_\mu$$

$$= -i \frac{\partial_\mu f}{f^2} u(-if u^+ D_\mu u) + i\frac{1}{f} D_\mu u \cdot (-if) u^+ D_\mu u$$

$$= -\frac{\partial_\mu f}{f} \cancel{u^+} D_\mu u + D_\mu u \cdot u^+ D_\mu u$$

$$= -\frac{\partial_\mu f}{f} D_\mu u - u D_\mu u^+ D_\mu u$$

$$= -\frac{1}{f}\left(\partial_\mu f \, D_\mu u + f u D_\mu u^+ D_\mu u\right)$$

$$= -\frac{1}{f} D_\mu (f u u^+) D_\mu u$$

$$D_\mu(\sqrt{f}\,u) = \partial_\mu\sqrt{f}\,u + \sqrt{f}\,D_\mu u \quad \rightarrow \cancel{D_\mu}$$

$$D_\mu D_\mu(\quad) = (\Box\sqrt{f})u + 2\partial_\mu\sqrt{f}\,D_\mu u + \sqrt{f}\,D_\mu D_\mu u$$

$$= \quad " \quad " \quad - \frac{1}{\sqrt{f}}\partial_\mu f\,D_\mu u - \sqrt{f}\,u\,\cancel{D_\mu D_\mu u}\; D_\mu u^+ D_\mu u$$

$$= (\Box\sqrt{f})u - \sqrt{f}\,u\,\cancel{D_\mu D_\mu u}\; D_\mu u^+ D_\mu u$$

$$\cancel{\Box\sqrt{f}\,u - \sqrt{f}\,u\,D_\mu}$$

$$u = \frac{1}{\sqrt{f}}U \qquad D_\mu u = \cancel{-\frac{1}{2}\frac{\partial_\mu f}{f^{3/2}}}\;\partial_\mu\left(\frac{1}{\sqrt{f}}\right)U + \frac{1}{\sqrt{f}}D_\mu U$$

$$\cancel{D_\mu D_\mu u = \Box\frac{1}{\sqrt{f}}}U +$$

$$= \Box\sqrt{f}\,u - U\left[\frac{1}{\sqrt{f}}D_\mu U^+ + \partial_\mu\left(\frac{1}{\sqrt{f}}\right)U^+\right]\left[\frac{1}{\sqrt{f}}D_\mu U + \partial_\mu\left(\frac{1}{\sqrt{f}}\right)U\right]$$

$$= \frac{\Box\sqrt{f}}{\sqrt{f}}U - \frac{1}{f}U D_\mu U^+ D_\mu U - U\left[\frac{1}{\sqrt{f}}\partial_\mu\left(\frac{1}{\sqrt{f}}\right)\{U^+ D_\mu U + D_\mu U^+\cdot U\}\right]$$

$$- UU^+U\;\partial_\mu\left(\frac{1}{\sqrt{f}}\right)\partial_\mu\left(\frac{1}{\sqrt{f}}\right)$$

$$= \frac{\Box\sqrt{f}}{\sqrt{f}}U - \frac{1}{f}U D_\mu U^+ D_\mu U - f\,U\,\partial_\mu\frac{1}{\sqrt{f}}\partial_\mu\frac{1}{\sqrt{f}} + \frac{1}{2}\frac{U}{f^2}\partial_\mu f\,\partial_\mu f$$

$$= \left(\frac{1}{2}\frac{1}{f}\Box f - \frac{1}{4}\frac{\partial_\mu f\,\partial_\mu f}{f^2}\right)U - \frac{1}{f}U D_\mu U^+ D_\mu U - \frac{1}{4}\frac{1}{f^2}\partial_\mu f\,\partial_\mu f + \frac{1}{2}\;"$$

$$= \left(\frac{1}{2}\frac{1}{f}\Box f - \cancel{\frac{1}{2}\frac{(\partial_\mu f)^2}{f^2}}\right)U - \frac{1}{f}U D_\mu U^+ D_\mu U$$

Suppose $Df = 0$. Then

$$f D_\mu D_\mu U + U D_\mu U^+ D_\mu U = 0$$

or $$\frac{1}{f} D_\mu D_\mu U + \frac{U}{f^2} D_\mu U^+ D_\mu U = 0$$

Let $$\mathcal{L} = F(U^+U) D_\mu U^+ D_\mu U$$

$$\rightarrow \quad D_\mu (F D_\mu U) - F' U D_\mu U^+ D_\mu U = 0$$

$$F D_\mu D_\mu U + F' \partial_\mu f D_\mu U - F' U D_\mu U^+ D_\mu U = 0$$

This consideration probably does not lead to anything. Individual U_n's are very simple $\sim x_n \cdot \sigma / x_n^2$. They cannot satisfy simple eqs involving D_μ since D_μ depends on all U_m's thru A_μ.

But we can construct a Lagrangian $\sim J_\mu J_\mu$

$$\sim U^+ \overleftrightarrow{D_\mu} U \; U^+ \overleftrightarrow{D_\mu} U \quad \text{or} \quad f^2 u^+ D_\mu u \; u^+ D_\mu u = -f^2 D_\mu u^+ D_\mu u$$

$$= -(U^+U)^2 (D_\mu \tfrac{1}{\sqrt{f}} U^+)(\partial_\mu \tfrac{1}{\sqrt{f}} U)$$

$$= -(U^+U)^2 \cancel{D_\mu U^+ D_\mu U} \left\{ \left(-\tfrac{1}{2} \partial_\mu f / f^{3/2}\right) U^+ + \tfrac{1}{\sqrt{f}} D_\mu U^+ \right\} \left\{ U^+ \to U \right\}$$

$$= -(U^+U)^2 \Big[\frac{1}{4} \frac{1}{(U^+U)^2} \partial_\mu (U^+U) \partial_\mu (U^+U) - \frac{1}{2} \frac{1}{f^2} (U^+ D_\mu U + D_\mu U^+ U) \partial_\mu f$$

$$+ \frac{1}{f} D_\mu U^+ D_\mu U \Big]$$

$$= -(U^+U)^2 \Big[\frac{1}{4} \frac{1}{(U^+U)^2} \partial_\mu (U^+U) \partial_\mu (U^+U) - \frac{1}{2} \frac{1}{(U^+U)^2} (\partial_\mu (U^+U))^2$$

$$+ \frac{1}{U^+U} D_\mu U^+ D_\mu U \Big]$$

So $L \sim - U^\dagger U \; D_\mu U^\dagger D_\mu U + \frac{1}{4} \partial_\nu (U^\dagger U) \partial_\mu (U^\dagger U) \sim \frac{1}{4} U^\dagger \overleftrightarrow{D}_\mu U \; U^\dagger \overleftrightarrow{D}_\mu U$

Her U is an $n \times m$ matrix.

A natural solution : $U^\dagger \overleftrightarrow{D}_\mu U = 0 , \quad D_\mu F_{\mu\nu} = 0$.

This formula not correct in general : f is not a unit matrix .

$$J_\mu J_\mu = -\mathrm{Tr}\tfrac{1}{4} (U^\dagger \overleftrightarrow{D}_\mu U)(U^\dagger \overleftrightarrow{D}_\mu U)$$

$$= -\frac{1}{4} \Big[U^\dagger D_\mu U \; U^\dagger D_\mu U - U^\dagger D_\mu U \, D_\mu U^\dagger U$$
$$- D_\mu U^\dagger U \, U^\dagger D_\mu U + D_\mu U^\dagger U \, D_\mu U^\dagger U \Big]$$

With our ansatz for U,

$$U^\dagger U = \sum_n f_n \, (1)_{2\times 2} \qquad \equiv f$$

$$(U U^\dagger)_{nm} = \cancel{2 \hat{x}_n \cdot \hat{x}_m \sqrt{f_n}\sqrt{f_m} \, (1)_{2\times 2}}$$
$$\left(\hat{x}_n \cdot \sigma \; \hat{x}_m \tilde{\sigma} \; \sqrt{f_n}\sqrt{f_m} \right)_{2\times 2}$$

$$D_\mu U^\dagger \cdot U = \partial_\mu f - U^\dagger D_\mu U \qquad J_\mu = -i (U^\dagger D_\mu U - \tfrac{1}{2} \partial_\mu f)$$

$$U^\dagger D_\mu U = \partial_\mu f - D_\mu U^\dagger U \qquad = i (D_\mu U^\dagger \cdot U - \tfrac{1}{2} \partial_\mu f)$$

So $J_\mu J_\mu = -\frac{1}{4} \Big[\partial_\mu f \; U^\dagger D_\mu U - D_\mu U^\dagger U \, U^\dagger D_\mu U - 2 \, U U^\dagger D_\mu U D_\mu U^\dagger$

$$+ D_\mu U^\dagger U \, \partial_\mu f - D U^\dagger U \, U^\dagger D_\mu U \Big]$$

$$= -\frac{1}{4} \Big[\partial_\mu f \, \partial_\mu f - 4 \, U U^\dagger D_\mu U D_\mu U^\dagger \Big]$$

$$= U U^\dagger D_\mu U D_\mu U^\dagger - \frac{1}{4} \partial_\mu f \, \partial_\mu f$$

$$\text{''} = D_\mu U^\dagger \cdot U U^\dagger D_\mu U - \cdots$$

Unless $UU^+ \propto 1_{n\times n}$, it does not make sense to divide it by UU^+. $\cancel{J_\mu J_\mu / UU^+}$

So this effective $\mathcal{L}$ is intrinsically nonlinear unfortunately.

Define another current $K_\mu = \frac{i}{2} U \overleftrightarrow{D}_\mu U^+$ $n\times n$

$$K_\mu K_\mu = -\frac{1}{4}\mathrm{Tr}\left[U D_\mu U^+ U D_\mu U^+ - U D_\mu U^+ D_\mu U U^+ \right.$$
$$\left. - D_\mu U U^+ U D_\mu U^+ + D_\mu U U^+ D_\mu U U^+ \right]$$
$$= \mathrm{Tr}\left(U^+U D_\mu D^+ D_\mu U - \frac{1}{4} D_\mu(UU^+)\, U D_\mu U^+ - \frac{1}{4} D_\mu(UU^+)\, D_\mu U U^+ \right)$$
$$= \mathrm{Tr}\left(U^+U D_\mu U^+ D_\mu U - \frac{1}{4} D_\mu(UU^+)\, D_\mu(UU^+) \right)$$

$$\delta(K_\mu K_\mu)/\delta A_\lambda = U^+ K_\lambda U = \frac{i}{2} U^+ U D_\mu U^+ U - \frac{i}{2} U^+ D_\mu U \cdot U^+ U$$
$$= \frac{i}{2} U^+U \left(D_\mu U^+ U - U^+ D_\mu U \right) \quad \text{if } U^+U \propto \mathbb{1}$$
$$= -U^+U J_\mu$$

Actually UU^+ is gauge independent, not U^+U.

So $K_\mu K_\mu + \frac{1}{4} D_\mu(UU^+) D_\mu(UU^+) = U^+U D_\mu U^+ D_\mu U$

has the same derivative w.r. to A_λ.

$$\delta J_\mu / \delta A_\lambda = U^+ U \, \delta_{\mu\lambda} \quad \rightarrow \quad \delta J_\mu J_\mu / \delta A_\lambda = \{ U^+ U J_\mu \}$$

$$\delta J_\mu J_\mu / \delta U^+ = -\tfrac{i}{2} D_\mu U J_\mu - \tfrac{i}{2} D_\mu (U J_\mu) \quad \rightarrow = 0 \text{ if } J_\mu = 0$$

$$\delta K_\mu K_\mu / \delta A_\lambda = - U^+ U J_\mu \quad \rightarrow 0$$

$$\delta K_\mu K_\mu / \delta U^+ = -\tfrac{i}{2} D_\mu (K_\mu U) - i K_\mu D_\mu U$$

$$= +\tfrac{1}{2} D_\mu (U \overleftrightarrow{D_\mu} U^+ \cdot U) + \tfrac{1}{2} (U \overleftrightarrow{D_\mu} U^+ D_\mu U) \quad \neq 0 .$$

If $J_\mu = 0 \rightarrow \quad U^+ D_\mu U = D_\mu U^+ \cdot U = \frac{1}{2} D_\mu (U^+ U)$

~~Then~~ $\cancel{D_\mu J_\mu = 0 = D_\mu D_\mu (U^+ U)}$

~~If~~ $\cancel{U^+ U = f = \sum \lambda_n / x_n^2 \mathbb{1} \rightarrow D_\mu f = \partial_\mu f \rightarrow \Box f = 0 \, ?}$

~~This is a bit suspicious~~

If $\quad U^+ U = f \mathbb{1} \quad D_\mu (U^+ U) = \partial_\mu f$

$D_\mu D_\mu (U^+ U) = 0 \rightarrow \Box f = 0 \quad$ except at sources.

Then $\quad D_\mu (U^+ D_\mu U) = D_\mu (D_\mu U^+ \cdot U) = 0 \quad$ "

or $\quad U^+ D_\mu^2 U = D_\mu^2 U^+ \cdot U$

$$A = \tfrac{1}{2} (U^+ D_\mu U + D_\mu U^+ U) = \mathrm{Re} \; U^+ D_\mu U$$

$$B = -\tfrac{i}{2} (U^+ D_\mu U - D_\mu U^+ U) = \mathrm{Im} \; U^+ D_\mu U$$

$$A^2 + B^2 = \tfrac{1}{2} (U^+ D_\mu U \, D_\mu U^+ U + D_\mu U^+ U \, U^+ D_\mu U)$$

Questions.

1. In electrodynamics, the Coulomb energy is the maximum of an action; not the minimum.

Either $$Z = \exp[-\tfrac{1}{2}E^2 + i\rho\phi]$$

or $$\exp[\tfrac{1}{2}E^2 - \cancel{i}\rho\phi].$$

2. The Wilson loop integral $\langle \exp[i\oint A\cdot dx]\rangle$ is not real for $SU(3)$. This is because $3 \not\simeq 3^*$.

Do Z and Z^* represent two different mesons?

Dec. 28, 78

A new ansatz for cylindrical symmetry

$$A_\mu = -if\, u^+ \partial_\mu u \quad \Rightarrow \quad u \to uv$$

$$\to -if\, \cancel{u^+ v}\, v^+ u^+ \partial_\mu u\, v - if\, v^+ \partial_\mu v$$

gauge transform by v:

$$\to -if\, u^+ \partial_\mu u - if\, v \partial_\mu v^+ - if\, \partial_\mu v\, v^+$$

$$= -if\, u^+ \partial_\mu u - i(1-f)\, v \partial_\mu v^+$$

Choose $\quad u = \exp[ikz^{\sigma_1}], \quad v = \exp[i\omega t^{\sigma_2}] \to \exp[-i\omega\tau]$

$$f = f(x, y)$$

$\tau = it$

$\omega = i\omega$

So $\quad A_3 = fk\sigma_1, \quad A_4 = (1-f)\omega\sigma_2$

$$A_1, A_2 = 0.$$

$$D_\mu = \partial_\mu - iA_\mu \; ; \quad F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$$F_{34} = -i[A_3, A_4] = f(1-f)\, k\omega\, \sigma_3$$

$$F_{12} = 0$$

$$F_{31} = -\partial_1 A_3 = -\partial_1 f\, k\sigma_1$$

$$F_{32} = -\partial_2 A_3 = -\partial_2 f\, k\sigma_1$$

$$F_{41} = -\partial_1 A_4 = -\partial_1 (1-f)\, \omega\sigma_2 = \partial_1 f\, \omega\sigma_2$$

$$F_{42} = \partial_2 f\, \omega\sigma_2$$

$$D_\mu F_{\mu\nu} =$$

$$D_\mu F_{\mu 4} = \partial_1 F_{14} + \partial_2 F_{24} + \partial_3 F_{34} - i [A_3, F_{34}]$$

$$= -\partial_1^2 f\, \omega\sigma_2 - \partial_2^2 f \omega\sigma_2 - i [f k \sigma_1, f(1-f) k\omega\sigma_3]$$

$$= -\nabla^2 f\, \omega\sigma_2 - f^2(1-f) k^2 \omega\, \sigma_2$$

$$\rightarrow \quad \nabla^2 f + f^2(1-f) k^2 = 0$$

$$D_\mu F_{\mu 3} = \partial_1 F_{13} + \partial_2 F_{23} + \partial_4 F_{43} - i [A_4, F_{43}]$$

$$= \partial_1^2 f\, k\sigma_1 + \partial_2^2 f\, k\sigma_1 + i [(1-f) \omega\sigma_2, f(1-f) k\omega\sigma_3]$$

$$= \nabla^2 f\, k\sigma_1 - f(1-f)^2 k\omega^2 \sigma_1$$

$$\rightarrow \quad \nabla^2 f - f(1-f)^2 \omega^2 = 0$$

There are not equal! Unfortunate.

$$D_\mu F_{\mu 1} = \cancel{\partial_2 F_{21} + \partial_3 F_{31} - i [A_3, F_{31}] + \partial_4 F_{41} - i [A_4, F_{41}]} = 0$$

$$D_\mu F_{\mu 2} = 0.$$

More generally, $\quad A_3 = f \sigma_1, \quad A_4 = g \sigma_2$

$$A_1, A_2 = 0.$$

$$F_{34} = fg\,\sigma_3$$

$$F_{3i} = -\partial_i f \sigma_1$$

$$F_{4i} = -\partial_i g\, \sigma_2$$

$$F_{12} = 0$$

Then the two eqs are (Minkowski)

$$\begin{cases} \nabla^2 g \mp g f^2 = 0 & \rightarrow \nabla^2 g + g f^2 = 0 \\ \nabla^2 f + f g^2 = 0 \end{cases}$$

Change $f \to if$ $\cancel{g \to ig}$ $\cancel{\nabla^2 g + f g^2 = 0}$.

There follow from $\mathcal{L} = \frac{1}{2}(\nabla f)^2 \mp \frac{1}{2}(\nabla g)^2 \mp \frac{1}{2} f^2 g^2$

In the original form, $\mathcal{L} = \frac{1}{2}(\nabla f)^2 \pm \frac{1}{2}(\nabla g)^2 \pm \frac{1}{2} f^2 g^2$

$$\begin{cases} g'' + \frac{1}{\rho} g' \mp g f^2 = 0 \\ f'' + \frac{1}{\rho} f' + f g^2 = 0 \end{cases}$$

$$\Rightarrow \quad g g'' + \frac{1}{\rho} g g' + f f'' + \frac{1}{\rho} f f' = 0$$

$$(f^2)' = 2ff', \quad (f^2)'' = 2ff'' + 2(f')^2 = 2f\left(-\frac{1}{\rho} f' + f g^2\right) + 2(f')^2$$

$$(f^2)'' + \frac{1}{\rho}(f^2)' = 2f^2 g^2 + 2(f')^2 \cancel{= \left[\frac{(f^2)'}{2f}\right]} = 2f^2 g^2 + 2\left(\frac{(f^2)'}{2f}\right)^2$$

But $f^2 = -\frac{1}{g}\left(g'' + \frac{1}{\rho} g'\right) = -\frac{1}{g\rho}(\rho g')'$

$(f^2)' =$

$$\begin{cases} g'' + \frac{1}{\rho} g' - g f^2 = 0 \\ f'' + \frac{1}{\rho} f' + f g^2 = 0 \end{cases}$$

If $f \sim \rho^n$, $g \sim \rho^m$ near $\rho = 0$.

f: $\quad n(n-1)\rho^{n-2} + n\rho^{n-2} + C'\rho^{n+2m} = 0$

g: $\quad m(m-1)\rho^{m-2} + m\rho^{m-2} - C\rho^{m+2n} = 0$

~~if $m > -1$ $\quad n^2 \rho^{n-2} = 0 \rightarrow n = 0$~~

~~Then $\quad m^2\rho^{m-2} + C'\rho^m = 0 \rightarrow C'$ must be $= 0$~~

For $f, g < \infty$ at $\rho = 0$, $\rightarrow n^2 = m^2 = 0$ so $f(0), g(0) \neq 0$

$f'(0) = g'(0) = 0$

and $f''(0)/f(0) < 0$, $\quad g''(0)/g(0) > 0$

At $\rho = \infty$, Can $g \to 0$, $f \to C$?

Then $\quad g'' + \frac{1}{\rho} g' + C^2 g \sim 0 \rightarrow g \sim J(\rho)$ oscillating and $\sim \frac{\cos(\cdot)}{\sqrt{\rho}}$

$\rightarrow \quad f'' + \frac{1}{\rho} f' - \frac{\cos(\,)}{\rho} f \sim 0$

If $g \to 0$, $f \to 0$ at $\rho = \infty$, we must have $m = -1$, $n = -1$

but also $n^2 + C' = 0 \quad m^2 - C = 0$

so $g \not\to 0$. $\rightarrow g \sim$ const $\rightarrow f \sim \frac{1}{\sqrt{\rho}} \exp(-\rho)$

$\rightarrow g \sim$ const $+ \cdots$
$\qquad\qquad \| $
$\qquad\qquad 1$

Check. $f'' + \frac{1}{\rho} f' + fC' \cong 0$ $\quad$ $g = $ Bessel fn $K_0 \sim \frac{1}{\sqrt{\rho}} e^{-\rho\sqrt{C}} \times A$

$g'' + \frac{1}{\rho} g' - gC \cong 0$ $\quad$ $f \sim C$ or $C' \ln \rho$ $\quad$ ~~∞~~, ~~$C=1$~~, $C' = 0$

Next order. $fg'' + \frac{1}{\rho} f g' + \frac{f}{g} g^2 = f'' + \frac{1}{\rho} f' + \underbrace{\frac{f}{g} \frac{e^{-2\rho\sqrt{C}}}{\rho^{1/2}}}_{C^2 A e^{-2\rho\sqrt{C}}/\rho^{1/2}} \simeq 0$

$f = f_0 + f_1 = C + f_1$, $\quad f_1'' + \frac{1}{\rho} f_1' = -AC\, e^{-2\sqrt{C}\rho}/\rho^{1/2}$

So $f_1 \sim \rho^k e^{-2\sqrt{C}\rho}$ $\quad f_1'' + \frac{1}{\rho} f_1' \sim 4C \rho^k e^{-2\sqrt{C}\rho}$ leading term.

$\rightarrow f_1 \sim -4A \rho^{-1/2} e^{-2\sqrt{C}\rho}$

Thus $\quad F_{34} \sim fg \sim CA\, e^{-\sqrt{C}\rho}/\sqrt{\rho}$

$F_{3i} \sim \partial_i f_1 \sim A\, e^{-2\sqrt{C}\rho}/\rho$

$F_{4i} \sim \partial_i g \sim A\, e^{-\sqrt{C}\rho}/\sqrt{\rho}$

At $\rho = 0$: ~~if singular (regular) $n = m = 1$ again $f \sim \rho^{-1}$, $g \sim 1$~~

f, g can be regular: $n = m = 0 \rightarrow f, g \rightarrow$ const $\neq 0$.

f, g can be $\sim \ln \rho$, $n = m = 0$

cannot be regular singular $n, m \neq 0$ because then, $m = -1$, $n = \pm\sqrt{C}$, $C = g(0)$ but $m = -1$ is not compatible

can be irregular singular? $n, m < -1$