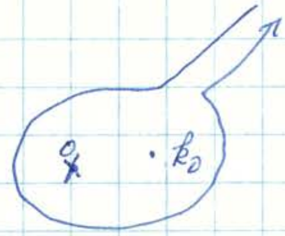
Scattering amplitude.

One vortex problem   Let's remove the i in exp:

$$\int_{\Gamma_1+\Gamma_2} \exp\left[\cancel{i}\left(kz + \frac{\bar{z}}{k}\right)\right] k^{\varkappa}/(k-k_0)$$

1. Γ_1 goes to ∞



$\varkappa < 0$ for convergence when $z \to 0$.

2. Γ_2 goes to 0

$\varkappa > -1$

Asymptotic form. $|z| \to \infty$

$$\left(z + \frac{\bar{z}}{k^2}\right) + \frac{\varkappa}{k} - \frac{1}{k-k_0} = 0, \qquad kz + \frac{\bar{z}}{k} + \varkappa - \frac{k}{k-k_0} = 0$$

As $z \to \infty$, $k - k_0 \to 0$.

So $\quad k_0 z + \frac{\bar{z}}{k_0} \approx \frac{k_0}{\varepsilon} \qquad \varepsilon = k - k_0$

~~k_0 — imag.;~~ $|k_0| = 1$ by assumption.

Second derivative

$$-2\frac{\bar{z}}{k^3} - \frac{\varkappa}{k^2} + \frac{1}{(k-k_0)^2} \approx -2\frac{\bar{z}}{k_0^3} + \frac{1}{\varepsilon^2}$$

$$\frac{1}{\varepsilon} = z + \frac{\bar{z}}{k_0^2} \quad \rightarrow \quad -2\frac{\bar{z}}{k_0^3} + \left(z + \frac{\bar{z}}{k_0^2}\right)^2$$

$$\sim \left(z + \frac{\bar{z}}{k_0^2}\right)^2 = \frac{1}{k_0^2}\left(k_0 z + \frac{\bar{z}}{k_0}\right)^2$$

The $\int$: $\sim \quad \exp[k_0 z - \frac{\bar{z}}{k_0}]\, k_0^{\kappa} \left(z + \frac{\bar{z}}{k_0^2}\right)$

$$\times \int \exp\left[\frac{1}{2k_0^2}\left(k_0 z + \frac{\bar{z}}{k_0}\right)^2 \cdot \xi^2\right] d\xi$$

Suppose $k_0 = \text{real} = 1$ (beam in y direction)

$\rightarrow \quad \frac{1}{\varepsilon} = z + \bar{z} = \text{real}$

$\int d\xi \rightarrow \sqrt{2\pi} / |z + \bar{z}| \times \text{sign}.$

Sign depends on the sign of $\varepsilon = k - k_0$

if $\varepsilon > 0$, the saddle pt belongs to Γ_1 with pos. sign of path

$\varepsilon < 0$ " Γ_2, neg. sign of path

So $|z + \bar{z}| \rightarrow z + \bar{z}$

and the whole $\int \rightarrow \exp[k_0 z - \frac{\bar{z}}{k_0}]\, k_0^{\kappa}$

This is the incident wave.

The scattered wave comes from ~~the next correction~~

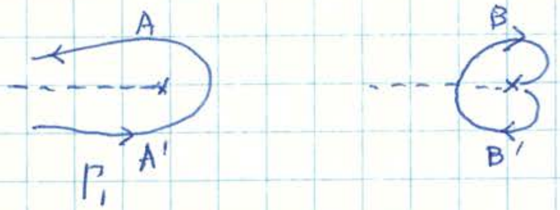
~~Or~~ k not near k_0.

$z + \bar{z}/k^2 \approx 0 \qquad k^2 \approx -\bar{z}/z \qquad k \sim \pm i (\bar{z}/z)^{1/2}$

$kz - \bar{z}/k \approx \pm 2i\cancel{k}(z\bar{z})^{1/2}$

Which sign?

The sign depends on which Riemann sheet one takes for Γ_1 & Γ_2.



If A & B are on the same sheet, they cancel. But A' & B' are on different sheets, $\rightarrow$ difference $e^{2\pi i \kappa} - 1$.

Now 2-vortex problem.

We have to maintain symmetry between 0 & ∞, so all the exp. factors should show it:

Assume : $$\exp\left[kz - \frac{\bar z}{k} - al + \frac{\bar a}{l}\right] k^{\varkappa_1}(k-l)^{\varkappa_{12}}\, l^{\varkappa_2}(l-l_0)^{-1}$$

Γ_1 $|k| > |l| > |l_0|$; Γ_2 $|k| < |l| < |l_0|$

$\varkappa_1 + \varkappa_{12} < -1$ from Γ_1 $\varkappa_1 > -1$ from Γ_2

Γ_1 :

Γ_2

$\cancel{\varkappa_1 > -1}$

Asymptotic form. First

k: $$z + \frac{\bar z}{k^2} + \frac{\varkappa_{12}}{k-l} + \frac{\varkappa_1}{k} = 0 \quad \to \quad kz + \frac{\bar z}{k} = -\frac{\varkappa k}{\varepsilon} \quad \to \quad k \sim l$$
$$k - l = \varepsilon$$

$\cancel{l: z - \frac{\bar a}{l^2} - \frac{\varkappa_{12}}{k-l} + \frac{\varkappa_2}{l} + \frac{1}{l-l_0} = 0}$ or $lz + \frac{\bar z}{l} \cong -\frac{\varkappa l}{\varepsilon}$

2nd derivative $$-\frac{2\bar z}{k^3} - \frac{\varkappa_{12}}{(k-l)^2} \sim -\frac{\varkappa_{12}}{\varepsilon^2} \approx -\left(z + \frac{\bar z}{l^2}\right)^2 \frac{1}{\varkappa_{12}}$$

$$\exp \to \exp\left[l(z-a) - \frac{\bar z - \bar a}{l}\right] \exp\left[-\frac{1}{2}\left(z + \frac{\bar z}{l^2}\right)^2 \frac{1}{\varkappa_{12}} \xi^2\right]$$
$$\times\ \varepsilon^{\varkappa_{12}}\, l^{\varkappa_2 + \varkappa_1}(l-l_0)^{-1}$$
$$\hookrightarrow \left[-\xi\left(z + \frac{\bar z}{l^2}\right)/\varkappa_{12}\right]^{-\varkappa_{12}}$$

$\int d\xi$ gives $$\exp\left[l(z-a) - \frac{\bar z - \bar a}{l}\right] \left[-\left(z + \frac{\bar z}{l^2}\right)/\varkappa_{12}\right]^{-\varkappa_{12}-1}$$
$$\times\ l^{\varkappa_2 + \varkappa_1}(l-l_0)^{-1}$$
$$= \exp\left[l(z-a) - \frac{\bar z - \bar a}{l}\right] \left(-\left(zl + \frac{\bar z}{l}\right)/\varkappa_{12}\right)^{-\varkappa_{12}-1}$$
$$\times\ l^{\varkappa_1 + \varkappa_2 + \varkappa_{12} + 1}(l-l_0)^{-1}$$

Next $\partial/\partial l$:

$$z-a+\frac{\bar{z}-\bar{a}}{l^2}+\frac{z-\frac{\bar{z}}{l^2}}{zl+\frac{\bar{z}}{l}}\times(-\varkappa_{12}-1)+\frac{\overbrace{\varkappa_1+\varkappa_{12}+1}^{+\varkappa_2}}{l}-\frac{1}{l-l_0}=0$$

Again $l\sim l_0$: $\quad z+\frac{\bar{z}}{l^2}-\frac{1}{\eta}\approx 0 \qquad \eta=l-l_0$

$$z+\frac{\bar{z}}{l_0^2}\approx\frac{1}{\eta}$$

We are left with a factor $\sim(z+\bar{z}/l_0^2)^{-\varkappa_{12}-1}$

To remove it, we should have used ~~the~~ a factor

$(l-l_0)^{-2-\varkappa_{12}}$ instead of $(l-l_0)^{-1}$. so

$$(k-l)^{\varkappa_{12}}\,l^{\varkappa_2}\,(l-l_0)^{-2-\varkappa_{12}}$$

Ambiguity in the choice of $\varkappa$'s.

$$k^{\varkappa_1}(k-l)^{\varkappa_{12}}\,l^{\varkappa_2}\,(l-l_0)^{-2-\varkappa_{12}}$$

$\varkappa_1+\varkappa_{12}<-1 \quad , \quad \varkappa_1>-1$

$\rightarrow \varkappa_{12}<0$

This makes it suspicious that one has the correct solution?

Both $k\to\infty$ & $k\to 0$ must have the same circulation.

So $\quad \varkappa_1+\varkappa_{12}=\varkappa_1+n \quad \rightarrow \underline{\varkappa_{12}=\text{integer}}$

Choose $\varkappa_{12}=-1$ Then the unpleasant factor goes away:

$\cancel{z+\bar{z}/l_0} \quad (l-l_0)^{-2-\varkappa_{12}}\rightarrow(l-l_0)^{-1}$

Thus the condition on $\varkappa_1$: $\boxed{-1 < \varkappa_1 < 0}$

This choice makes it simpler to formulate recurrence relations.

$$f_{n+1}(l) = \int^{\infty} f_n(l')\, e^{-\left(l' a_{n+1} - \frac{\bar{a}_{n+1}}{l'}\right)} l'^{\varkappa} / (l'-l)\, dl' \qquad |l'| > l$$

with
$$f_0(l) = \int^{\infty} e^{\left(kz - \frac{\bar{z}}{k}\right)} k^{\varkappa_0} / (k-l)\, dk \qquad |k| > l$$

Similarly define $\tilde{f}_n$ with contours around 0..

The scat. amplitude will be $f_n(l) + \tilde{f}_n(l)$, with l being the incident momentum.

The f's are entire functions.

$$f_n(l) = \sum_{i=0}^{\infty} C_i^{(n)} l^i$$

$$f_{n+1}(l) = \sum_{i,m} \int \left(C_i^{(n)} l'^{\,i+\varkappa-m-1} l^m \right) e^{-\left(l' a_{n+1} - \frac{\bar{a}_{n+1}}{l'}\right)} dl'$$

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} C_i^{(n)} J_{-i-\varkappa+m}\big(\underset{\downarrow\ \sqrt{2}?}{|a|}\big) \left(\frac{\bar{a}_{n+1}}{a_{n+1}}\right)^{\frac{i+\varkappa-m}{2}} l^m$$

This is singular as $a \to 0$.

$$C_m^{(n+1)} = \sum_{i=0}^{\infty} C_i^{(n)} J_{-i-\varkappa+m}(|a|_{n+1}) \left(\frac{\bar{a}_{n+1}}{a_{n+1}}\right)^{\frac{i+\varkappa-m}{2}}$$

Now turn to $\tilde{f}$: Transf. $l \to -1/l$

$$\exp[l'a - \frac{\bar{a}}{l'}]\, l'^{\varkappa}/(l'-l)\, dl'$$

$$\to \exp[\bar{a}l' - \frac{a}{l'}]\, l'^{-\varkappa}/(l'-l) \cdot l\, l'^{-1} dl'$$

To keep the symmetry, I could have defined

$$f_{n+1} = \int \exp[l'a - \frac{\bar{a}}{l'}]\, l'^{\varkappa} (l'/l)^{-1/2}/(l'-l)\, dl' \; f_n$$

Then under $l, l' \to -1/l, -1/l'$,

$a \leftrightarrow \bar{a}$, $\varkappa \leftrightarrow -\varkappa$ Circulation is $\varkappa - 1/2$

$-\frac{1}{2} < \varkappa < 1/2$

Our previous formulas for f should now read

$$f_n = l^{1/2} \sum C_i^{(n)} l_i^n$$

no change otherwise.

$\tilde{f}_n$ is obtained by $\varkappa \leftrightarrow -\varkappa$, $a \to \bar{a}$.

This amounts to:

$$(\bar{a}/a)^{\frac{i+\varkappa-m}{2}} \to (a/\bar{a})^{\frac{i-\varkappa-m}{2}} = (\bar{a}/a)^{\frac{-i+\varkappa+m}{2}}$$

or $i, m \to -i, -m$.

Problem: $f_{0,-1}$ does not fit into this form

7/31 Asymptotic form

f_0: $$z + \frac{\bar{z}}{k^2} + \frac{\varkappa}{k} - \frac{1}{k-\ell} = 0$$

$$\rightarrow \quad k - \ell = \varepsilon \qquad \frac{1}{\varepsilon} \cong z + \frac{\bar{z}}{\ell^2} + \frac{\varkappa}{\ell}$$

$$\text{or} \quad z + \frac{\bar{z}}{\ell^2}$$

2nd der. $$-\frac{2\bar{z}}{k^3} - \frac{\varkappa}{k^2} + \frac{1}{(k-\ell)^2} \rightarrow \frac{1}{\varepsilon^2}$$

$$f_0 \sim \exp\left(\ell z - \frac{\bar{z}}{\ell}\right) \ell^{\varkappa} \underbrace{\left(z + \frac{\bar{z}}{\ell^2}\right)}_{\varepsilon} \int \exp\left(\tfrac{1}{2}\varepsilon^2 \zeta^2\right) d\zeta$$

$$\Rightarrow \exp\left(\ell z - \frac{\bar{z}}{\ell}\right) \ell^{\varkappa} \sqrt{2\pi} \quad (+ \text{ scattered wave})$$

So $$f_1 \sim \int \exp\left[(z-a)\ell_1^* - \frac{\bar{z}-a}{\ell_1^*}\right] \ell^{\varkappa_0+\varkappa_1} / (\ell_1^* - \ell_2)\, d\ell_1^*$$

$$\Rightarrow \exp\left[\ell_2 z - \frac{\bar{z}}{\ell_2}\right] \ell^{\varkappa_0+\varkappa_1} (\sqrt{2\pi})^2, \quad \text{etc.}$$

$$\times \exp(-a\ell_2 + \bar{a}/\ell_2)$$

Examine the magnitudes of k, ℓ'..

Both $\sim z + \bar{z}/\ell^2$

$$\frac{1}{k-\ell_1} = \frac{1}{\varepsilon} = z + \frac{\bar{z}}{k^2} + \frac{\varkappa_0}{k} \approx z + \frac{\bar{z}}{\ell_1^2} - \frac{2\bar{z}\varepsilon}{\ell_1^3} = z + \frac{\bar{z}}{\ell_1^2} - \frac{2\bar{z}}{z + \frac{\bar{z}}{\ell_1^2}} \frac{1}{\ell_1^3} + \frac{\varkappa_0}{\ell_1}$$

$$\frac{1}{\ell_1 - \ell_2} = \frac{1}{\eta} = z - a + \frac{\bar{z}-a}{\ell_1^2} + \frac{\varkappa_1}{\ell_1}$$

Well: $$\frac{\ell_1}{k-\ell_1} \cong z\ell_1 + \frac{\bar{z}}{\ell_1}$$

$$\frac{\ell_2}{\ell_1-\ell_2} \cong z\ell_2 + \frac{\bar{z}}{\ell_2}$$

First scattered wave:

$$z + \frac{\bar z}{k^2} \approx 0 \qquad k = (-\bar z/z)^{1/2} \equiv k_0, \quad |k_0| = 1$$

if $k \neq l$

$$kz + \bar z/k \approx 2(-z\bar z)^{1/2}$$

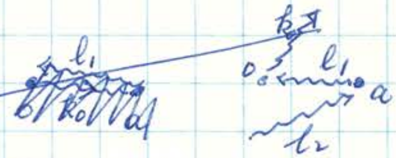
2nd der. $\approx -2\bar z/k^3 = -2\bar z\,(-\bar z/z)^{-3/2} = 2(z/\bar z)\,(-z\bar z)^{1/2}$

$$f_0^{sc.} \sim \exp[2(-z\bar z)^{1/2}]\ \underbrace{(-\bar z/z)^{1/2}}_{= k_0}{}^{\times \varkappa_0}\,(k_0 - l_1)^{-1} \cdot (z/\bar z)^{-1/2}\,(-z\bar z)^{-1/4}$$

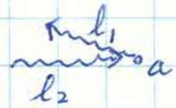
We have to integrate: $\displaystyle\int \frac{1}{k_0 - l_1}\, l_1^{\varkappa_1}\, \frac{1}{l_1 - l_2}\, e^{-al_1 + \bar a/l_1}\, dl_1$

$$\frac{1}{k_0 - l_1}\,\frac{1}{l_1 - l_2} = \left(\frac{1}{k_0 - l_1} + \frac{1}{l_1 - l_2}\right) / (k_0 - l_2)$$

~~1st term: double scat~~



~~2nd term:~~



Next $f_1^{sc.}$ comes from $z - a + \frac{\bar z - \bar a}{l^2} \sim 0$

~~Scattered by a single source of strength $\varkappa_0 + \varkappa_1$~~

~~Meaning of f_0^{sc} not clear.~~

f_0^{sc} scattered last from source o, f_1^{sc} from source 1. ?

f_0^{sc} contains terms $l_1 \approx k_0$ & $l_1 \approx l_2$:

$$\sim \frac{1}{k_0 - l_2}\left(e^{-ak_0 + \frac{\bar a}{k_0}} - e^{-al_2 + \frac{\bar a}{l_2}}\right)$$

This looks like a phase difference bet inc. & ~~out~~ scat. waves at pt $-a$. Or does only one of them contribute ?.

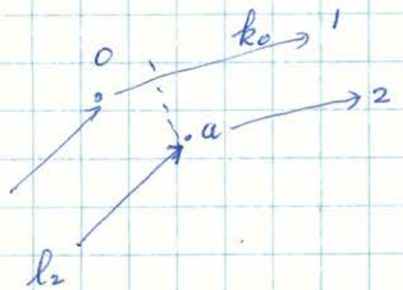
The incoming wave in f_1 has phase factors associated with pt $z-a$ and vorticity $\varkappa_0+\varkappa_1$, so the scat waves also are referred to pt $z-a$.

Thus, f_1^{sc} has a phase determined by $z-a/\bar{z}-\bar{a}$ and $\varkappa_0+\varkappa_1$; f_0^{sc} has $\varkappa_0\,(z/\bar{z})$ ~~and $\varkappa_1\,(z a/\bar{z}\bar{a})$~~

and an extra phase factor

$$(*) \qquad \cancel{\exp} \left(e^{-ak_0+\frac{\bar{a}}{k_0}}_{\widehat{k_0\ \varkappa_1}} - e^{-al_2+\frac{\bar{a}}{l_2}}_{\widehat{l_2\ \varkappa_1}} \right)$$

This factor is a bit strange.



phase difference between 1 & 2:

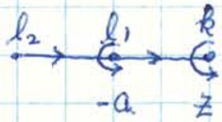
$$\cancel{e^{a(k_0-l_2)-\bar{a}/(k_0-l_2)}}$$

$$e^{ak_0-\bar{a}/k_0}\ e^{-ak_2+\bar{a}/l_2}$$

The second term in $(*)$ removes from f_1^{sc} the part which comes from source 0 with phase $\varkappa_0$, and ~~the~~ replaces it with the first term

Attempt to construct the correct scattering solution.

One criterion is that for zero flux it reduces to a plane wave. Let's combine two solutions centered around ∞ and 0. For two vortices, we have the representation

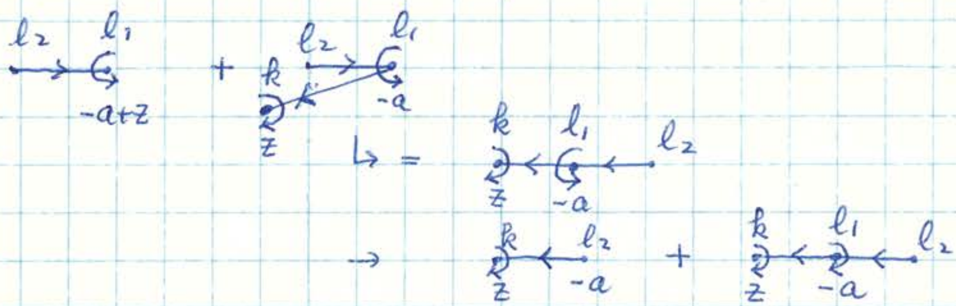


Indicates

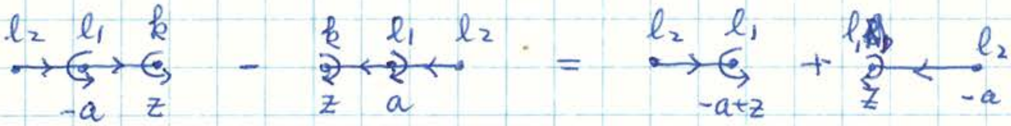
$$\iint \frac{1}{l_2-l_1} e^{-al_1+\frac{\bar a}{l_1}} \cdot \frac{1}{l_1-k} e^{\frac{\bar z}{\bar a}k - \frac{z}{k}} \, dl_1 \, dk_1$$

C means going to ∞ (to the right) with $|l_2| < |l_1| < |k|$ ~~indicated by the arrow~~.

Move the contours to the right thus:



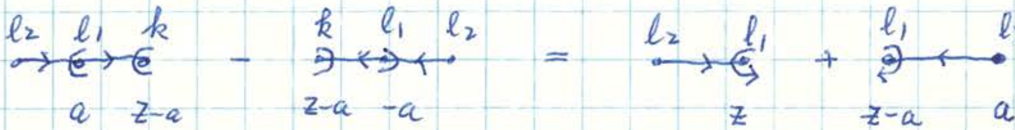
So



I

~~Re~~ Change of order : $z \to z-a$, $-a \to +a$

Then



II

Take $\mathrm{I} - \mathrm{II} \exp[-al_2 + \frac{\bar a}{l_2}]$

sums up to

$= 0$?

So $\mathrm{I} \approx \mathrm{II}$. We cannot have a plane wave left over.

Take I alone.

$$\int_0 \frac{dl_1}{l_2 - l_1} e^{(z-a)l_1 - \frac{\bar{z}-\bar{a}}{l_1}} \qquad |l_2| > |l_1|$$

This has plane + scattered $\rightsquigarrow$ both outgoing & incoming
around $z = a$ $\qquad l_1 = \pm\left(-\bar{z}+\bar{a}/z-a\right)^{1/2}$

The other integral

$$\int_0^\infty \frac{dl_1}{l_2 - l_1} e^{z l_1 - \frac{\bar{z}}{l_1}} \; e^{-a l_2 + \frac{\bar{a}}{l_2}}$$

contains plane + scattered centered around z

1. These two scat. waves do not cancel
2. How to ~~make~~ keep only outg. or incom. part?

Go back to one-vortex problem. Again here the two Sols have phases which are designed to cancel one of them.

For the 2-vortex problem, we need cancellations for two scattered waves $\longrightarrow$ we need 4 independent solutions.

But we have only 2 so far. This must be at the root of the problem.

Should we use more general flux factors like

$$k^{\varkappa_0}(k-l)^{\varkappa_{01}} \quad \text{etc ?}$$

Or: Interchange the positions z, $z-a$ vs $\bar{z}-\bar{a}$, $\bar{z}$?

$$\exp\left[zk - \frac{\bar{z}-\bar{a}}{k} - l_1 a - \frac{\bar{a}}{l_1}\right]$$

$$= \exp\left[z(k-l) + l_1(z-a) - (\bar{z}-\bar{a})\left(\frac{1}{k} - \frac{1}{l_1}\right) - \bar{z}/l_1\right]$$

Flux factors $\quad k^{\varkappa_0}(k-l_1)^{\varkappa_{01}} l_1^{\varkappa_1}(l_1-l_2)^{\varkappa_{12}}$

around z: $\varkappa_0 + \varkappa_{01}$

$z-a$: $\varkappa_1$

$\bar{z}-\bar{a}$: $\varkappa_0$ $\quad \rightarrow \quad \varkappa_0 = \varkappa_1$

$\bar{z}$: $\varkappa_{01}+\varkappa_1$

Also we need $\quad \varkappa_{01} + \varkappa_{12} = -2$

So we write

$$k^{\varkappa_0}(k-l_1)^{-1+\mu} l_1^{\varkappa_0}(l_1-l_2)^{-1-\mu}$$

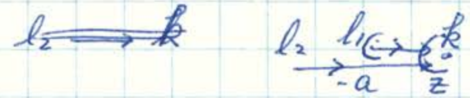
circulation around z : $\varkappa_0 + \mu \quad$ (mod n)

" $\quad z-a$: $\varkappa_0$

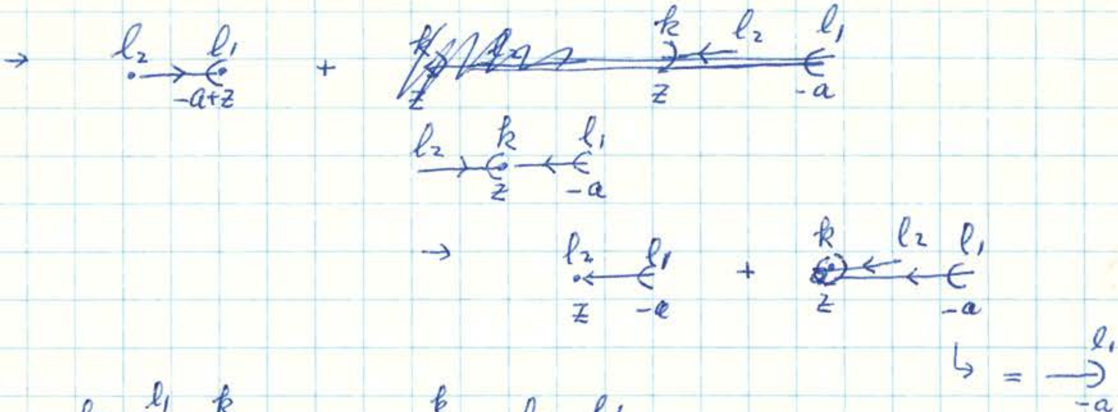
This function has a crazy ~~but~~ property: Coordinates centered at z and $\bar{z}-\bar{a}$

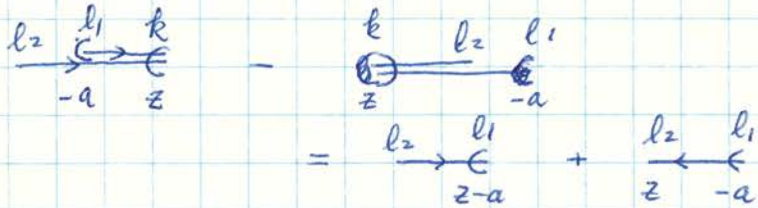
But it does not seem helpful.

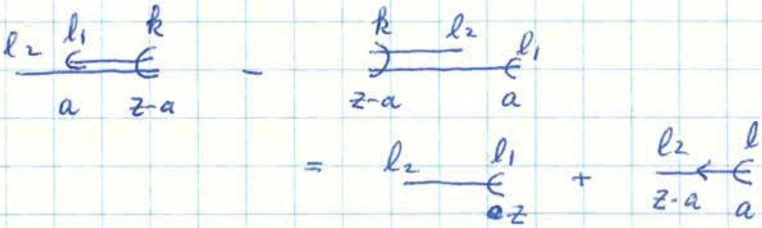
Another function.

Lets link : 

This is admissible. Now shift :



So  III

and  IV

Combining with I, III + IV gives 

which is a plane wave.

Q: The difference of I & III is a fn; ~~sum of $f(z)$ & $g(z-a)$~~

$$\frac{1}{l_2-l_1}\,\frac{1}{l_1-k} - \frac{1}{l_2-k}\,\frac{1}{l_1-k} = \frac{1}{l_2-l_1}\,\frac{1}{l_2-k}$$

so it leads to a product fn $f(z)\,g(-a)$

which has no branch at $z=a$. Is it possible?

The branch indices must be multiplicative:

$$\underset{\bar z}{F(z)} = z^{\alpha}(z-a)^{\beta}\,G\left(\substack{\bar z\\ z}\right)$$

↳ no br. pts (meromorphic)

No. It had the form $z^{\alpha}(z-a)^{\beta}\,G + \bar z^{-\alpha}(\bar z-\bar a)^{-\beta}H$.

Yes: there can be fns $\sim z^{\alpha}(\bar z-\bar a)^{-\beta}$, $\bar z^{-\alpha}(z-a)^{\beta}$

<u>These can be product fns.</u>

Thus we do not have to use the fns III & IV;

instead use forms etc.

?

But this yields a fn $\sim z^{\alpha}a^{\beta}$?

Something must be wrong.

Set $l_2 = 0$.

$$-\frac{1}{l_1}\frac{1}{l_1-k} + \frac{1}{k}\frac{1}{l_1-k} = \frac{1}{l_1}\frac{1}{k}$$

Each term on left $\sim \sum \left(\frac{z}{\bar z}\right)^{\nu/2} J_\nu(\sqrt{z\bar z}) \left(\frac{a}{\bar a}\right)^{-\nu/2} J_{-\nu'}(\sqrt{a\bar a})$

ν, ν' differ by an integer.

This is supposed to sum into a form $z^\nu (z-a)^\mu \times$ entire fn.

But ~~that is~~ the r.h.s. is a single term. $\cancel{J_\varkappa J_\mu J_\varkappa} \sim J_\varkappa J_\mu$ //

Furthermore, l.h.s. integral depends on $|k| > |l|$ or $|k| < |l|$

$$J_\nu(z) \sim \frac{1}{\nu!}|z|^\nu \quad \text{etc for large } \nu.$$

$$\sum \sim \sum \frac{1}{(n+\varkappa)!} z^{n+\varkappa} \frac{1}{(-n+\lambda)!} a^{-n+\lambda} \sim \frac{1}{(\lambda+\varkappa)!}\left(1+\frac{z}{a}\right)^{\lambda+\varkappa} z^\varkappa a^\lambda$$

So it creates a factor $(z+a)^{\lambda+\varkappa}$

To be exact $-\frac{1}{l_1}\frac{1}{l_1-k} = \sum_{n=0}^{\infty} \frac{l_1^{n-1}}{k^{n+1}} \to J_{n+\varkappa}(|z|)\left(\frac{z}{\bar z}\right)^{\frac{n+\varkappa}{2}} J_{-n+\lambda}(|a|)\left(\frac{a}{\bar a}\right)^{\frac{-n+\lambda}{2}}$

Similarly $-\frac{1}{k}\frac{1}{l_1-k} = \sum \frac{l_1^n}{k^{n+2}} \to J_{n+\varkappa+1} \times J_{-n-1+\lambda}$

It leads to the same $\sim \left(1+\frac{z}{a}\right)^{\lambda+\varkappa} \cdot z^{\varkappa+1} a^{\lambda-1}$

This shows only that the fn has a part ~~that~~ with the branch factor. Why?

~~On~~ Examine the proof of the existence of singularity.

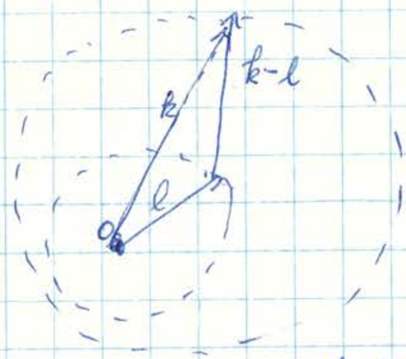
1. Pinch due to $1/(k-l)$

2. Corresponding exponent factor

$$\exp[kz - la] : \rightarrow (k-l)z + (z-a)l$$

If $k-l$ & l are independent, one needs $\mathrm{Re}(k-l)z < 0$
$\mathrm{Re}(z-a)l < 0$

But they ~~cannot~~ are not indep because they have to encircle 0 . ?



Yes they do encircle 0.

For this, $|k-l| > |l|$

$$k-l \equiv \xi$$

$$\int \exp[\xi z + (z-a)l]\ \xi^{x_0} l^{\lambda} = ? \qquad |\xi| > |l| \rightarrow \xi + l \neq 0$$

$$+\frac{\bar{z}}{\xi+l} \qquad \frac{\bar{z}}{k} - \frac{\bar{a}}{l} = \bar{z}$$

This also develops singularity as $\xi + l \rightarrow 0$, which we avoid.

The real problem: the inevitavility of rotating the phases of k & l with z & $z-a$. The restriction $|\xi| > |l|$ can spoil it.

Expansion around $\bar{z} \sim 0$:

$$\exp(\bar{z}/_{\xi+\ell}) \to \sum (\bar{z}/_{\xi+\ell})^n/n!$$

$\int \exp(\xi z)\, \xi^{\varkappa-n} \sim z^{n-\varkappa}$ from dimensionality alone.

$$\int \exp[(z-a)\ell - \bar{a}/\ell]\, \ell^{m-\lambda} \to J_{\lambda-m+1}$$

Problem: index around $z \sim 0$ no problem, but around $z \sim a$.

$J_{\lambda-m-1} \to (z-a)^{-\frac{\lambda-m-1}{2}}$ blows up.

Or $\int \exp(\xi z)\, \xi^{\varkappa}/(\xi+\ell)^n \, d\xi = \cdots$

$$\int \frac{\exp(\xi z)\exp((z-a)\ell)\, d\ell\, d\xi}{(\xi+\ell)^n}\ \xi^{\varkappa}\ell^{\lambda} \qquad |\xi|>|\ell|$$

At $z=a$: $$\int e^{\xi a}\frac{\xi^{\varkappa}\ell^{\lambda}}{(\xi+\ell)^n}\, d\ell\, d\xi \qquad |\xi|>|\ell|$$

The ℓ-integration up to $|\ell| = |\xi|$:

~~$\xi = \ell\xi$~~

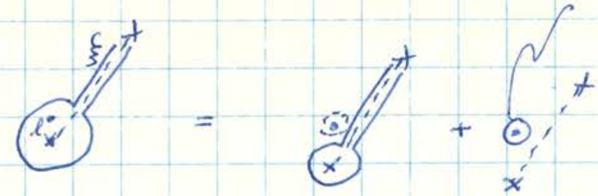
$\ell = \xi\varsigma, \quad |\varsigma| < 1$

$$\int \frac{\ell^{\lambda}}{(\xi+\ell)^n}\, d\ell \to \xi^{\lambda+1-n}\int \frac{\varsigma^{\lambda}}{(1+\varsigma)^n}\, d\varsigma$$

$\hookrightarrow$ 0_+  up to some pt ς_0, $|\varsigma_0| = 1$, $\neq -1$

Then integrate $\int d\xi\, e^{\xi a}\, \xi^{\lambda+1-n} \propto a^{n-\lambda-1}$

The result depends on ς_0 : no good!!! Baka.



$$\frac{\exp(\xi z + l(z-a))}{\xi + l}\, \xi^{\kappa} l^{\lambda}$$

$|\xi| > |l|$ Here $|\xi| < |l|$, $|\xi| = |l|$ substitution

Expand $1/_{\xi+l} = \sum (-\xi)^n / l^{n+1}$ $\exp[(z-a)\xi l + \bar{a} l]\, l^{\kappa+\lambda}$

so we have terms $\sim (z-a)^{-\kappa}$ $= \exp(\bar{z} l)\, l^{\kappa+\lambda}$

$\to$ const.

Does not work for the strip 

Thus there is one term with wrong phase.

Then do we have to use the contour l_0  ?

~~Interchange the roles of ξ & l, we get a term $\propto (z-a)^{-\kappa-\lambda-1}$~~

$$\int e^{\xi z}\, e^{l(z-a)}\, \xi^{\kappa} l^{\lambda} / (l+\xi) \qquad |\xi| > |l| \qquad \text{good}$$

$$\frac{1}{\xi}\sum_{n=0}^{\infty}\left(-\frac{l}{\xi}\right)^n$$

$$\int^{\infty} e^{\xi z}\, \xi^{-n+\kappa}\, d\xi = c\, z^{n-\kappa+1} / \Gamma(n-\kappa)$$

$$\to \int = \sum_{n=0}^{\infty} (-1)^n z^{n-\kappa+1} / \Gamma(n-\kappa+1) \cdot (z-a)^{-n-\lambda} / \Gamma(-n-\lambda)$$

$$\cancel{= \frac{1}{\Gamma(1-\lambda-\kappa)}\, z^{-1+\kappa} (z-a)^{\lambda} (1 - z/(z-a))^{-1-\lambda-\kappa}}$$

2

Operate with $\frac{\partial}{\partial z} + \frac{\partial}{\partial(z-a)}$: $\sum (-1)^n z^{n-\kappa} / \Gamma(n-\kappa)\ (z-a)^{-n-\lambda} / \Gamma(-n-\lambda)$

$+ \sum + (-1)^n z^{n-\kappa+1} / \Gamma(n-\kappa+1)\ (z-a)^{-n-\lambda-1} / \Gamma(-n-\lambda-1)$

= o.k. only the 1st term survives.

So unless something unusual happens to the series, this has the right factors $z^{-\varkappa}(z-a)^{-\lambda}$. But as $z-a=0$, it $\to\infty$. ?

Not if $\lambda<0$: Choose the path $l\to\infty$ = path $\xi\to\infty$.

In fact, $\to 0$. Same is true when $z=0$. if $\varkappa<0$.

So we will revise our prescription:

~~$\exp[k_0 z_0 + k_2 z_2 =$~~

$$\iint \exp\left[kz + l(z-a) - \frac{\bar z}{k+l} + \frac{\bar a}{l}\right] k^{\varkappa} l^{\lambda}\, dk\, dl \qquad |k|>|l|$$

~~corresponding to~~ In our earlier notation:

$$\iint \exp\left[kz - \frac{\bar z}{k} - al - \frac{\bar a}{l}\right](k-l)^{\varkappa} l^{\lambda}\, dk\, dl \qquad |k|>|l|$$

But the asymptotic form has a problem.

What's wrong with the previous one: $k^{\varkappa} l^{\lambda} (k-l)^{-1}$?

or in the present notation, a factor

$$\int e^{\xi z} e^{l(z-a)} \xi^{-1} l^{\lambda}/(l+\xi)^{-\varkappa}$$

Still, it can be expanded $\sum (l/\xi)^n c_n \xi^{\varkappa-1} l^{\lambda}$

With ξ^{-1} removed, $\to \int e^{(\xi+l)z - la} l^{\lambda} f(l+\xi)^{\varkappa}$

Does that cause a factorization? Not if ξ & l are chosen as variables constrained by $|\xi|>|l|$.

$$(l+\xi)^{\varkappa} = \xi^{\varkappa}\sum_{n=0}^{\infty}\binom{\varkappa}{n\ \ \varkappa-n}(l/\xi)^{n}$$

$$\int e^{\xi z}\,\xi^{-n+\varkappa}\,d\xi = z^{n-\varkappa\mp1}/\Gamma(n-\varkappa)$$

$$\int e^{l(z-a)}\,l^{n+\lambda}\,dl = (z-a)^{-n-\lambda\mp1}/\Gamma(-n-\lambda)$$

So we have $$\sum_{n=0}^{\infty}\underbrace{\frac{\varkappa!}{n!\,(\varkappa-n)!}\,\frac{1}{(n-\varkappa-1)!}}\,\frac{1}{(-n-\lambda-1)!}\,\frac{z^{n-\varkappa\mp1}}{(z-a)^{n+\lambda+1}}$$

$$\Gamma(\varkappa-n+1)\,\Gamma(n-\varkappa) = \frac{\pi}{\sin\pi(n-\varkappa)}$$

$$\rightarrow \sum\frac{\varkappa!}{n!\,(-n-\lambda-1)!}\,\frac{\pi}{\sin\pi(n-\varkappa)}\,\frac{z^{n-\varkappa\mp1}}{(z-a)^{n+\lambda+1}}$$

$$= \sum\frac{\varkappa!}{n!}\,(n+\lambda)!\,\frac{\sin\pi(-n-\lambda)}{\pi}\,\frac{\sin\pi(n-\varkappa)}{\pi}\,\frac{z^{n-\varkappa\mp1}}{(z-a)^{n+\lambda+1}}$$

$$= \sum\frac{(-\lambda-1)!}{n!\,(-n-\lambda-1)!}\,\frac{z^{n-\varkappa+1}}{(z-a)^{n+\lambda-1}}\,\frac{\varkappa!}{(-\lambda-1)!}\,(-1)^{n}\,\frac{\sin\pi(n-\varkappa)}{\pi}$$

$\hookrightarrow (-1)^{n+1}\sin\varkappa$

$$= \left(1-\frac{z}{z-a}\right)^{-\lambda-1}\cdot z^{-\varkappa\mp1}(z-a)^{-\lambda\mp1}\,\frac{\varkappa!}{(-\lambda-1)!}\left(\frac{\sin\pi\varkappa}{\pi}\right)$$

$$= (-a)^{-\lambda-1}\,\cancel{(z-a)}\,z^{-\varkappa+1}\,\frac{\varkappa!}{(-\lambda-1)!}\left(-\frac{\sin\pi\varkappa}{\pi}\right)$$ 0 No!

$$\sin\pi(n-\varkappa) = \left(e^{\pi i n-\pi i\varkappa} - e^{-\pi i n+i\pi\varkappa}\right)/2i \qquad e^{\pi i n} = (-1)^{n}$$

~~$\rightarrow \sum = 1$~~

If we insert ξ^{-1} in the $\int$al? It corresponds to $\int_{z}^{\infty}dz$ with $z-a$ fixed. But not convergent unless ~~$\varkappa$ sufficiently large $\to 0$~~ ~~or $\lambda \to 0$~~ $\lambda-\varkappa < -1$

Or else $$\int e^{\xi z}\,\xi^{-n+\varkappa-1}\,d\xi \rightarrow z^{n-\varkappa}/\Gamma(n-\varkappa+1)$$

This changes the series to:

$$\sum \frac{(-\lambda-1)!}{n!\,(-n-\lambda-1)!}\;\frac{z^{n-\varkappa}}{(z-a)^{n+\lambda-1}}\;\frac{1}{n-\varkappa}\cdot\frac{\varkappa!}{(-\lambda-1)!}\,(-1)^{n+1}\,\frac{\sin\varkappa\pi}{\pi}$$

corresponding to $\int_0^z dz$ of the original $\int$al. But this would not produce a new branch pt at $z=a$.?

As $z \to a$: $\left(1-\frac{\xi}{z-a}\right)^{-\lambda-1}(z-a)^{-\lambda+1}\,\xi^{-\varkappa+1} \to [\xi-(z-a)]^{-\lambda-1}\,\xi^{-\varkappa+1}$

Expansion in $\xi/z-a$ when $|\xi|<|z-a|$

$z-a/\xi$ $\quad|\xi|>|z-a|$

Add a factor ξ^{μ} instead: $z^{n-\varkappa-\mu-1}/\Gamma(n-\varkappa-\mu)$

$(n-\varkappa-\mu-1)!\,(-n-\lambda-1)!$ Let $\varkappa+\mu+\lambda=-1$

$\to \Gamma(n-\varkappa-\mu)\,\Gamma(-n-\lambda) \to \pi/\sin\pi(-n-\lambda)$ This is no good.?

$$(-n-\lambda-1)! = \Gamma(-n-\lambda) = \pi/\sin\pi(-n-\lambda)\,\Gamma(1+n+\lambda) = -\pi/\sin(n+\lambda)\cdot(n+\lambda)!$$

$$(\varkappa-n)! = \Gamma(\varkappa-n+1) = \pi/\sin\pi(n-\varkappa)\,\Gamma(n-\varkappa)$$

$$\sum = \sum \frac{\varkappa!}{n!\,(\varkappa-n)!}\,\frac{(n+\lambda)!}{(n-\varkappa-\mu-1)!}\,\frac{z^{n-\varkappa-\mu-1}}{(z-a)^{n+\lambda+1}}\,\frac{-\pi}{\sin(n+\lambda)}$$

$$\cancel{\sum \Gamma(\varkappa+1)} = \Gamma(\varkappa+1)\sum\frac{\Gamma(n+\lambda+1)\,\Gamma(n-\varkappa)}{\Gamma(n+1)\,\Gamma(n-\varkappa-\mu)}\,\frac{z^{n-\varkappa-\mu-1}}{(z-a)^{n+\lambda+1}}\,\frac{\pi^2}{\sin\pi\varkappa\,\sin\pi\lambda}$$

$$= \Gamma(\varkappa+1)\,{}_2F_1\left(\lambda+1,-\varkappa;-\varkappa-\mu;\frac{z}{z-a}\right)\frac{z^{-\varkappa-\mu-1}}{(z-a)^{\lambda+1}}\cdot\frac{\Gamma(\lambda+1)\,\Gamma(-\varkappa)}{\Gamma(-\varkappa-\mu)}\,\frac{\pi^2}{\sin\pi\varkappa\,\sin\pi\lambda}$$

Index of F at $1-\frac{z}{z-a}=\frac{-a}{z-a}$. Or as $\frac{z}{z-a}\to\infty$,

It has 2 parts: $\sim\left(\frac{z}{z-a}\right)^{-\lambda-1}$ & $\left(\frac{z}{z-a}\right)^{\varkappa}$

which lead to $z^{-\lambda-\varkappa-\mu-2}$, $z^{-\mu-1}(z-a)^{-\varkappa-\lambda-1}$ resp. $\left\{\begin{array}{l}\mu \text{ fixed}\\ \varkappa+\lambda+1 \text{ fixed mod. } n\end{array}\right.$

To cancel the 1st: $\lambda+\varkappa+\mu+1=0$ or positive int ?

Then at $z\approx 0$, $\sim z^{\lambda}$; at $z=a$, $\sim(z-a)^{\mu}$

So one should consider $\quad n \geq 0$

$$\int e^{\xi z + \ell(z-a)} \xi^{\mu} \ell^{\lambda} / (\ell+\xi)^{\mu+\lambda+1-n} \, d\xi \, d\ell, \quad |\xi| > |\ell|$$

problem: cannot be used for $\exp[\bar{z}/(\ell+\xi)]$

~~Another possibility:~~ Another way: let $\Gamma(-\kappa) \to \infty$.

Again κ = positive or $(\ell+\xi)^{\kappa>0}$ no good.

We are thus stuck with a ~~fo~~ part which does not behave right.

~~then~~ Then cancellation after summing? Or contour integration w.r. to the parameters? Neither one works.